Combinations

Detailed Examples

Main Ideas

- Combinations are as easy to setup as permutations. There are extra steps, but the patterns are similar.
- The hard part of calculating combinations by hand is canceling the numbers in the bottom of the fraction (the denominator).

Example 7.9

Compute the following:

1. ${}_8C_3$
2. ${}_{12}C_5$
3. ${}_{15}C_9$

Step	Work
Write out the permutation ${}_8P_3$ in the top of the fraction.	$\frac{8 \times 7 \times 6}{\quad}$

Write out 3! in the bottom of the fraction.

$$\frac{8 \times 7 \times 6}{3 \times 2 \times 1}$$

Cancel the 3 in the bottom by dividing it into the 6 in the top. 6 divided by 3 is 2. The 2 stays in the top of the fraction.

$$\frac{8 \times 7 \times \cancel{6}^{\,2}}{\cancel{3}_{\,1} \times 2 \times 1} \qquad 6 \div 3 = 2$$

Cancel the 2 in the bottom by dividing it into the 8 in the top. 8 divided by 2 is 4. The 4 stays in the top of the fraction.

There are only 1's left in the bottom. That means we can just ignore it.

$$\frac{{}^{4}\cancel{8} \times 7 \times \cancel{6}^{\,2}}{\cancel{3}_{\,1} \times \cancel{2}^{\,1} \times 1} \qquad \begin{array}{l} 6 \div 3 = 2 \\ 8 \div 2 = 4 \end{array}$$

Multiply $4 \cdot 7 \cdot 2 = 56$.

$$\frac{\overset{4}{\cancel{8}} \times 7 \times \overset{2}{\cancel{6}}}{\overset{1}{\cancel{3}} \times \overset{1}{\cancel{2}} \times 1}$$

$6 \div 3 = 2$

$8 \div 2 = 4$

$$4 \times 7 \times 2 = 56$$

Our answer is ${}_8C_3 = 56$.

Step	Work
Write out the permutation ${}_{12}P_5$ in the top of the fraction.	$\frac{12 \times 11 \times 10 \times 9 \times 8}{}$
Write out 5! in the bottom of the fraction.	$\frac{12 \times 11 \times 10 \times 9 \times 8}{5 \times 4 \times 3 \times 2 \times 1}$

Cancel the 5 in the bottom by dividing it into the 10 in the top. 10 divided by 5 is 2. The 2 stays in the top of the fraction.

$$\frac{12 \times 11 \times \overset{2}{\cancel{10}} \times 9 \times 8}{\overset{1}{\cancel{5}} \times 4 \times 3 \times 2 \times 1} \qquad 10 \div 5 = 2$$

Cancel the 4 in the bottom by dividing it into the 12 in the top. 12 divided by 4 is 3. The 3 stays in the top of the fraction.

$$\frac{\overset{3}{\cancel{12}} \times 11 \times \overset{2}{\cancel{10}} \times 9 \times 8}{\overset{1}{\cancel{5}} \times \cancel{4}^{\,1} \times 3 \times 2 \times 1} \qquad \begin{array}{l} 10 \div 5 = 2 \\ 12 \div 4 = 3 \end{array}$$

Cancel the 3 in the bottom by dividing it into the 9 in the top. 9 divided by 3 is 3. The 3 stays in the top of the fraction.

$$\frac{\overset{3}{\cancel{12}} \times 11 \times \overset{2}{\cancel{10}} \times \cancel{9}^{\,3} \times 8}{\overset{1}{\cancel{5}} \times \cancel{4}^{\,1} \times \cancel{3}^{\,1} \times 2 \times 1} \qquad \begin{array}{l} 10 \div 5 = 2 \\ 12 \div 4 = 3 \\ 9 \div 3 = 3 \end{array}$$

Cancel the 2 in the bottom by dividing it into the 8 in the top. 8 divided by 2 is 4. The 4 stays in the top of the fraction.

There are only 1's left in the bottom. That means we can just ignore it.

$$\frac{\overset{3}{\cancel{12}} \times 11 \times \overset{2}{\cancel{10}} \times \overset{3}{\cancel{9}} \times \overset{4}{\cancel{8}}}{\underset{1}{\cancel{5}} \times \overset{1}{\cancel{4}} \times \overset{1}{\cancel{3}} \times \overset{1}{\cancel{2}} \times 1}$$

$10 \div 5 = 2$
$12 \div 4 = 3$
$9 \div 3 = 3$
$8 \div 2 = 4$

Multiply $3 \cdot 11 \cdot 2 \cdot 3 \cdot 4 = 792$.

$$\frac{\overset{3}{\cancel{12}} \times 11 \times \overset{2}{\cancel{10}} \times \overset{3}{\cancel{9}} \times \overset{4}{\cancel{8}}}{\underset{1}{\cancel{5}} \times \overset{1}{\cancel{4}} \times \overset{1}{\cancel{3}} \times \overset{1}{\cancel{2}} \times 1}$$

$10 \div 5 = 2$
$12 \div 4 = 3$
$9 \div 3 = 3$
$8 \div 2 = 4$

$3 \times 11 \times 2 \times 3 \times 4$
$= 792$

Our answer is ${}_{12}C_5 = 792$.

Step	Work
Write out the permutation ${}_{15}P_9$ in the top of the fraction.	$\frac{15 \times 14 \times 13 \times 12 \times 11 \times 10 \times 9 \times 8 \times 7}{}$

Write out 9! in the bottom of the fraction.

$$\frac{15 \times 14 \times 13 \times 12 \times 11 \times 10 \times 9 \times 8 \times 7}{9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}$$

Cancel out 9, 8, and 7 from the denominator with the matching 9, 8, and 7 from the numerator.

$$\frac{15 \times 14 \times 13 \times 12 \times 11 \times 10 \times \cancel{9} \times \cancel{8} \times \cancel{7}}{\cancel{9} \times \cancel{8} \times \cancel{7} \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}$$

Cancel the 6 in the bottom by dividing it into the 12 in the top. 12 divided by 6 is 2. The 2 stays in the top of the fraction.

$$\frac{15 \times 14 \times 13 \times \cancel{12}^{2} \times 11 \times 10 \times \cancel{9} \times \cancel{8} \times \cancel{7}}{\cancel{9} \times \cancel{8} \times \cancel{7} \times \cancel{6} \times 5 \times 4 \times 3 \times 2 \times 1}$$

$12 \div 6 = 2$

Cancel the 5 in the bottom by dividing it into the 10 in the top. 10 divided by 5 is 2. The 2 stays in the top of the fraction.

$$\frac{15 \times 14 \times 13 \times 12^{2} \times 11 \times 10^{2} \times 9 \times 8 \times 7}{9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}$$

$12 \div 6 = 2$

$10 \div 5 = 2$

Cancel the 4 in the bottom by dividing it into the two 2's in the top.

$$\frac{15 \times 14 \times 13 \times 12^{2} \times 11 \times 10^{2} \times 9 \times 8 \times 7}{9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}$$

$12 \div 6 = 2$

$10 \div 5 = 2$

$(2 \times 2) \div 4 = 1$

Cancel the 3 in the bottom by dividing it into the 15 in the top. 15 divided by 3 is 5. The 5 stays in the top of the fraction.

$$\frac{5\ 15 \times 14 \times 13 \times 12^{2} \times 11 \times 10^{2} \times 9 \times 8 \times 7}{9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}$$

$12 \div 6 = 2$

$10 \div 5 = 2$

$(2 \times 2) \div 4 = 1$

$15 \div 3 = 5$

Cancel the 2 in the bottom by dividing it into the 14 in the top. 14 divided by 2 is 7. The 7 stays in the top of the fraction.

There are only 1's left in the bottom. That means we can just ignore it.

$$\frac{\cancel{15}^{5} \times \cancel{14}^{7} \times 13 \times \cancel{12}^{\cancel{2}} \times 11 \times \cancel{10}^{\cancel{2}} \times \cancel{9} \times \cancel{8} \times \cancel{7}}{\cancel{9} \times \cancel{8} \times \cancel{7} \times \cancel{6} \times \cancel{5} \times \cancel{4} \times \cancel{3} \times \cancel{2} \times 1}$$

$12 \div 6 = 2$
$10 \div 5 = 2$
$(2 \times 2) \div 4 = 1$
$15 \div 3 = 5$
$14 \div 2 = 7$

Multiply $5 \cdot 7 \cdot 13 \cdot 11 = 5005$.

$$\frac{\cancel{15}^{5} \times \cancel{14}^{7} \times 13 \times \cancel{12}^{\cancel{2}} \times 11 \times \cancel{10}^{\cancel{2}} \times \cancel{9} \times \cancel{8} \times \cancel{7}}{\cancel{9} \times \cancel{8} \times \cancel{7} \times \cancel{6} \times \cancel{5} \times \cancel{4} \times \cancel{3} \times \cancel{2} \times 1}$$

$5 \times 7 \times 13 \times 11 = 5005$

$12 \div 6 = 2$
$10 \div 5 = 2$
$(2 \times 2) \div 4 = 1$
$15 \div 3 = 5$
$14 \div 2 = 7$

Our answer is ${}_{19}C_{9} = 5005$.