7.6 Probability with Permutations and Combinations

Figure 7.32 Bingo and many lottery games depend on selecting one or more numbers at random from a list; often this is done by drawing numbered balls from a bin. (credit: "Redundant Bingo Balls" by Greg Clarke/Flickr, CC BY 2.0)

Learning Objectives

After completing this section, you should be able to:

1. Calculate probabilities with permutations.
2. Calculate probabilities with combinations.

In our earlier discussion of theoretical probabilities, the first step we took was to write out the sample space for the experiment in question. For many experiments, that method just isn't practical. For example, we might want to find the probability of drawing a particular 5-card poker hand. Since there are 52 cards in a deck and the order of cards doesn't matter, the sample space for this experiment has ${}_{52}C_5 = 2{,}598{,}960$ possible 5-card hands. Even if we had the patience and space to write them all out, sorting through the results to find the outcomes that fall in our event would be just as tedious.

Luckily, the formula for theoretical probabilities doesn't require us to know every outcome in the sample space; we just need to know how many outcomes there are. In this section, we'll apply the techniques we learned earlier in the chapter (The Multiplication Rule for Counting, permutations, and combinations) to compute probabilities.

Using Permutations to Compute Probabilities

Recall that we can use permutations to count how many ways there are to put a number of items from a list in order. If we're looking at an experiment whose sample space looks like an ordered list, then permutations can help us to find the right probabilities.

EXAMPLE 7.23

Using Permutations to Compute Probabilities

1. In horse racing, an exacta bet is one where the player tries to predict the top two finishers in particular race in order. If there are 9 horses in a race, and a player decided to make an exacta bet at random, what is the probability that they win?
2. You are in a club with 10 people, 3 of whom are close friends of yours. If the officers of this club are chosen at random, what is the probability that you are named president and one of your friends is named vice president?
3. A bag contains slips of paper with letters written on them as follows: A, A, B, B, B, C, C, D, D, D, D, E. If you draw 3 slips, what is the probability that the letters will spell out (in order) the word BAD?

Solution

1. Since order matters for this situation, we'll use permutations. How many different exacta bets can be made? Since there are 9 horses and we must select 2 in order, we know there are ${}_9P_2 = 56$ possible outcomes. That's the size of our sample space, so it will go in the denominator of the probability. Since only one of those outcomes is a winner, the numerator of the probability is 1. So, the probability of randomly selecting the winning exacta bet is $\frac{1}{56}$.
2. There are 10 people in the club, and 2 will be chosen to be officers. Since the order matters, there are ${}_{10}P_2 = 90$ different ways to select officers. Next, we must figure out how many outcomes are in our event. We'll use the Multiplication Rule for Counting to find that number. There is only 1 choice for president in our event, and there are 3 choices for vice president. So, there are $1 \times 3 = 3$ outcomes in the event. Thus, the probability that you will serve as president with one of your friends as vice president is $\frac{3}{90} = \frac{1}{30}$.
3. There are 12 slips of paper in the bag, and 3 will be drawn. So, there are ${}_{12}P_3 = 1320$ possible outcomes. Now, we'll compute the number of outcomes in our event. The first letter drawn must be a B, and there are 3 of those. Next must come an A (2 of those) and then a D (4 of those). Thus, there are $3 \times 2 \times 4 = 24$ outcomes in our event. So, the probability that the letters drawn spell out the word BAD is $\frac{24}{1320} = \frac{1}{55}$.

YOUR TURN 7.23

1. Another bag of letters contains C, C, C, C, D, D, I, I, I, T, T, T, Y, Y, Y, Y. What is the probability that 4 letters chosen at random will spell, in order, CITY?

Combinations to Computer Probabilities

If the sample space of our experiment is one in which order doesn't matter, then we can use combinations to find the number of outcomes in that sample space.

EXAMPLE 7.24

Using Combinations to Compute Probabilities

1. Palmetto Cash 5 is a game offered by the South Carolina Education Lottery. Players choose 5 numbers from the whole numbers between 1 and 38 (inclusive); the player wins the jackpot of $100,000 if the randomizer selects those numbers in any order. If you buy one ticket for this game, what is the probability that you win the top prize by choosing all 5 winning numbers?
2. There's a second prize in the Palmetto Cash 5 game that a player wins if 4 of the player's 5 numbers are among the 5 winning numbers. What's the probability of winning the second prize?
3. *Scrabble* is a word-building board game. Players make hands of 7 letters by selecting tiles with single letters printed on them blindly from a bag (2 tiles have nothing printed on them; these blanks can stand for any letter). Players use the letters in their hands to spell out words on the board. Initially, there are 100 tiles in the bag. Of those, 44 are (or could be) vowels (9 As, 12 Es, 9 Is, 8 Os, 4 Us, and 2 blanks; we'll treat Y as a consonant). What is the probability that your initial hand has no vowels?

Solution

1. There are 38 numbers to choose from, and the order of the 5 we pick doesn't matter. So, there are ${}_{38}C_5 = 501,492$ outcomes in the sample space. Only one outcome is in our winning event, so the probability of winning is $\frac{1}{501,492}$.
2. As in part 1 of this example,, there are 501,492 outcomes in the sample space. The tricky part here is figuring out how many outcomes are in our event. To qualify, the outcome must contain 4 of the 5 winning numbers, plus one losing number. There are ${}_5C_4 = 5$ ways to choose the 4 winning numbers, and there are $38 - 5 = 33$ losing numbers. So, using the Multiplication Rule for Counting, there are $5 \times 33 = 165$ outcomes in our event. Thus, the probability of winning the second prize is $\frac{165}{501,492} = \frac{55}{167,164}$, which is about 0.00033.
3. The number of possible starting hands is ${}_{100}C_7 = 16,007,560,800$. There are $100 - 44 = 56$ consonants in the bag, so the number of all-consonant hands is ${}_{56}C_7 = 231,917,400$. Thus, the probability of drawing all consonants is $\frac{231,917,40}{16,007,560,800} = \frac{32,139}{2,425,388} \approx 0.0145$.

YOUR TURN 7.24

1. At a charity event with 58 people in attendance, 3 raffle winners are chosen. All receive the same prize, so order doesn't matter. You are attending with 4 of your friends. What is the probability that at least one of you or your friends wins a raffle prize? Hint: Find the probability that none of you wins, and use the formula for complements.
2. If you draw a hand of 5 cards from a standard deck, what is the probability that 2 cards are ♠ and 3 cards are ♡?

Check Your Understanding

For the following exercises, you are drawing *Scrabble* tiles without replacement from a bag containing the letters A, C, E, E, I, N, N, S, S, W.

31. What is the probability that you draw (in order) the letters W-I-N?
32. What is the probability that you draw (in order) the letters W-I-S-E?
33. What is the probability that you draw (in order) the letters S-E-E-N?
34. What is the probability that you draw (in any order) the letters W-I-N?
35. What is the probability that you draw (in any order) the letters W-I-S-E?
36. What is the probability that you draw (in any order) the letters S-E-E-N?

SECTION 7.6 EXERCISES

The following exercises deal with our version of the game blackjack. In this card game, players are dealt a hand of two cards from a standard deck. The dealer's cards are dealt with the second card face up, so the order matters; the other players' hands are dealt entirely face down, so order doesn't matter. The goal of the game is to build a hand whose point value is as close as possible to 21 without going over. The point values of each card are as follows: numbered cards are worth the number on the face (for example, 8♣ is worth 8 points); jacks, queens, and kings are each worth 10 points, and aces are worth either 1 or 11 points (the player can choose). Players whose hands are worth less than 21 points may ask to be dealt additional cards one at a time until they either go over 21 points or they choose to stop.

1. What is the probability that a player (not the dealer) is dealt an initial hand worth 21 points? This can only happen with an ace and a card worth 10 points (10, J, Q, or K).
2. What is the probability that the dealer is dealt an initial hand worth 21 points, with an ace showing?
3. What is the probability that a player is dealt 2 cards worth 10 points each?
4. What is the probability that a player is dealt an initial hand with an 8 and a 3?
5. What is the probability that a player is dealt an initial hand with two 8s?
6. What is the probability that a player is dealt 2 ♡?
7. In some versions of the game, a player wins automatically if they draw a hand of 5 cards that doesn't go over 21 points. One way this can happen is if they draw 5 cards, all of which are A, 2, 3, or 4. What is the probability of drawing 5 cards from that collection?

In horse racing, a trifecta bet is one where the player tries to predict the top three finishers in order. In the following exercises, find the probability of choosing a winning trifecta bet at random when the field contains the given number of horses.

8. 6 horses
9. 8 horses
10. 10 horses

In the following exercises, you are about to draw *Scrabble* tiles from a bag without replacement; the bag contains the letters A, A, C, E, E, E, L, L, N, O, R, S, S, S, T, X.

11. What is the probability of drawing the letters E-A-R, in order?
12. What is the probability of drawing the letters E-A-R, in any order?
13. What is the probability of drawing the letters S-E-A-L, in order?
14. What is the probability of drawing the letters S-E-A-L, in any order?
15. What is the probability of drawing the letters L-A-S-S, in order?
16. What is the probability of drawing the letters L-A-S-S, in any order?
17. What is the probability of drawing 3 tiles that are all vowels?
18. What is the probability of drawing 3 tiles that are all consonants?
19. What is the probability of drawing 4 tiles in the pattern vowel-consonant-vowel-consonant, in order?

20. What is the probability of drawing 2 vowels and 2 consonants, in any order?
21. What is the probability of drawing at least 1 vowel when drawing four tiles? (Hint: use the Formula for Complements.)
22. What is the probability of drawing at least 1 consonant when drawing four tiles?

The following exercises involve the board game *Clue*, which involves a deck of 21 cards: 6 suspects, 6 weapons, and 9 rooms. At the beginning of the game, 1 card of each of the 3 types is secretly removed from the deck (the object of the game is to identify those 3 cards). The remaining 18 cards are dealt out to the players. Assuming there are 3 players, each player gets 6 cards. Find the probabilities of a player being dealt hands with the given characteristics.

23. All 6 cards are rooms.
24. 5 cards are suspects (the sixth can be anything).
25. None of the cards are rooms.
26. None of the cards are suspects.
27. 3 cards are suspects and 3 are weapons.
28. There are 2 cards of each type.
29. There are 3 rooms, 2 suspects, and 1 weapon.
30. There are 4 rooms and 5 suspects.

7.7 What Are the Odds?

Figure 7.33 Scratch-off lottery tickets, as well as many other games, represent the likelihood of winning using odds. (credit: “My Scratch-off Winnings” by Shoshanah/Flickr, CC BY 2.0)

Learning Objectives

After completing this section, you should be able to:

1. Compute odds.
2. Determine odds from probabilities.
3. Determine probabilities from odds.

A particular lottery instant-win game has 2 million tickets available. Of those, 500,000 win a prize. If there are 500,000 winners, then it follows that there are 1,500,000 losing tickets. When we evaluate the risk associated with a game like this, it can be useful to compare the number of ways to win the game to the number of ways to lose. In the case of this game, we would compare the 500,000 wins to the 1,500,000 losses. In other words, there are 3 losing tickets for every winning ticket. Comparisons of this type are the focus of this section.

Computing Odds

The ratio of the number of equally likely outcomes in an event E to the number of equally likely outcomes *not* in the event E' is called the **odds for** (or **odds in favor** of) the event. The opposite ratio (the number of outcomes not in the event to the number in the event E' to the number in the event E is called the **odds against** the event.