11. ${}_{12}P_{10}$
12. ${}_{14}P_{10}$
13. ${}_{10}P_{8}$
14. ${}_{15}P_{11}$

The following exercises are about the card game euchre, which uses a partial standard deck of cards: It only has the cards with ranks 9, 10, J, Q, K, and A for a total of 24 cards. Some variations of the game use the 8s or the 7s and 8s, but we'll stick with the 24-card version.

15. A euchre hand contains 5 cards. How many ways are there to receive a 5-card hand (where the order in which the cards are received matters, i.e., 9♡, J♡, K♣, 9♠, 10♠ is different from 9♠ J♡, 9♡, K♣, 10♠?
16. After all 4 players get their hands, the remaining 4 cards are placed facedown in the center of the table. How many arrangements of 4 cards are there from this deck?
17. Euchre is played with partners. How many ways are there for 2 partners to receive 5-card hands (where the order in which the cards are received matters)?
18. How many different arrangements of the full euchre deck are possible (i.e., how many different shuffles are there)?

The following exercises involve a horse race with 13 entrants.

19. How many possible complete orders of finish are there?
20. An exacta bet is one where the player tries to predict the top two finishers in order. How many possible exacta bets are there for this race?
21. A trifecta bet is one where the player tries to predict the top three finishers in order. How many possible trifecta bets are there for this race?
22. A superfecta bet is one where the player tries to predict the top four finishers in order. How many possible superfecta bets are there for this race?

7.3 Combinations

Figure 7.8 Combinations help us count things like the number of possible card hands, when the order in which the cards were drawn doesn't matter. (credit: "IMG_3177" by Zanaca/Flickr, CC BY 2.0)

Learning Objectives

After completing this section, you should be able to:

1. Distinguish between permutation and combination uses.
2. Compute combinations.
3. Apply combinations to solve applications.

In Permutations, we studied permutations, which we use to count the number of ways to generate an ordered list of a given length from a group of objects. An important property of permutations is that the order of the list matters: The results of a race and the selection of club officers are examples of lists where the order is important. In other situations, the order is not important. For example, in most card games where a player receives a hand of cards, the order in which the cards are received is irrelevant; in fact, players often rearrange the cards in a way that helps them keep the cards organized.

Combinations: When Order Doesn't Matter

In situations in which the order of a list of objects doesn't matter, the lists are no longer permutations. Instead, we call them **combinations**.

EXAMPLE 7.8

Distinguishing Between Permutations and Combinations

For each of the following situations, decide whether the chosen subset is a permutation or a combination.

1. A social club selects 3 members to form a committee. Each of the members has an equal share of responsibility.
2. You are prompted to reset your email password; you select a password consisting of 10 characters without repeats.
3. At a dog show, the judge must choose first-, second-, and third-place finishers from a group of 16 dogs.
4. At a restaurant, the special of the day comes with the customer's choice of 3 sides taken from a list of 6 possibilities.

Solution

1. Since there is no distinction among the responsibilities of the 3 committee members, the order isn't important. So, this is a combination.
2. The order of the characters in a password matter, so this is a permutation.
3. The order of finish matters in a dog show, so this is a permutation.
4. A plate with mashed potatoes, peas, and broccoli is functionally the same as a plate with peas, broccoli, and mashed potatoes, so this is a combination.

YOUR TURN 7.8

Decide whether the following represent permutations or combinations:

1. On Halloween, you give each kid who comes to your door 3 pieces of candy, taken randomly from a candy dish.
2. Your class is going on a field trip, but there are too many people for one vehicle. Your instructor chooses half the class to take the first vehicle.

Counting Combinations

Permutations and combinations are certainly related, because they both involve choosing a subset of a large group. Let's explore that connection, so that we can figure out how to use what we know about permutations to help us count combinations. We'll take a basic example. How many ways can we select 3 letters from the group A, B, C, D, and E? If order matters, that number is ${}_5P_3 = 60$. That's small enough that we can list them all out in the table below.

ABC	ABD	ABE	ACB	ACD	ACE
ADB	ADC	ADE	AEB	AEC	AED
BAC	BAD	BAE	BCA	BCD	BCE
BDA	BDC	BDE	BEA	BEC	BED
CAB	CAD	CAE	CBA	CBD	CBE
CDA	CDB	CDE	CEA	CEB	CED
DCA	DAC	DAE	DBA	DBC	DBE
DCA	DCB	DCE	DEA	DEB	DEC
EAB	EAC	EAD	EBA	EBC	EBD
ECA	ECB	ECD	EDA	EDB	EDC

Now, let's look back at that list and color-code it so that groupings of the same 3 letters get the same color, as shown in Figure 7.9:

ABC	ABD	ABE	ACB	ACD	ACE
ADB	ADC	ADE	AEB	AEC	AED
BAC	BAD	BAE	BCA	BCD	BCE
BDA	BDC	BDE	BEA	BEC	BED
CAB	CAD	CAE	CBA	CBD	CBE
CDA	CDB	CDE	CEA	CEB	CED
DAB	DAC	DAE	DBA	DBC	DBE
DCA	DCB	DCE	DEA	DEB	DEC
EAB	EAC	EAD	EBA	EBC	EBD
ECA	ECB	ECD	EDA	EDB	EDC

Figure 7.9

After color-coding, we see that the 60 cells can be seen as 10 groups (colors) of 6. That's no coincidence! We've already seen how to compute the number of permutations using the formula To compute the number of combinations, let's count them another way using the Multiplication Rule for Counting. We'll do this in two steps:

Step 1: Choose 3 letters (paying no attention to order).

Step 2: Put those letters in order.

The number of ways to choose 3 letters from this group of 5 (A, B, C, D, E) is the number of combinations we're looking for; let's call that number ${}_5C_3$ (read "the number of combinations of 5 objects taken 3 at a time"). We can see from our chart that this is ten (the number of colors used). We can generalize our findings this way: remember that the number of permutations of n things taken r at a time is ${}_nP_r = \frac{n!}{(n-r)!}$. That number is *also* equal to ${}_nC_r \times r!$, and so it must be the case that $\frac{n!}{(n-r)!} = {}_nC_r \times r!$. Dividing both sides of that equation by $r!$ gives us the formula below.

FORMULA

$${}_nC_r = \frac{n!}{r!(n-r)!}$$

EXAMPLE 7.9

Using the Combination Formula

Compute the following:

1. ${}_8C_3$
2. ${}_{12}C_5$
3. ${}_{15}C_9$

Solution

1. ${}_8C_3 = \frac{8!}{3!(8-3)!} = \frac{8 \times 7 \times \mathbf{6} \times \mathbf{5!}}{\mathbf{3} \times \mathbf{2} \times \mathbf{1} \times \mathbf{5!}} = 8 \times 7 = 56$
2. ${}_{12}C_5 = \frac{12!}{5!(12-5)!} = \frac{\mathbf{12} \times 11 \times \mathbf{10} \times 9 \times 8 \times \mathbf{7!}}{\mathbf{5} \times \mathbf{4} \times \mathbf{3} \times \mathbf{2} \times \mathbf{1} \times \mathbf{7!}} = 11 \times 9 \times 8 = 792$
3. ${}_{15}C_9 = \frac{15!}{9!(15-9)!} = \frac{\mathbf{15} \times 14 \times 13 \times \mathbf{12} \times 11 \times 10 \times \mathbf{9!}}{\mathbf{9!} \times \mathbf{6} \times \mathbf{5} \times 4 \times \mathbf{3} \times \mathbf{2} \times 1} = \frac{14 \times 13 \times 11 \times 10}{4} = 5{,}005$

YOUR TURN 7.9

Compute the following:

1. $_{6}C_{4}$
2. $_{10}C_{8}$
3. $_{14}C_{5}$

EXAMPLE 7.10

Applying the Combination Formula

1. In the card game Texas Hold'em (a variation of poker), players are dealt 2 cards from a standard deck to form their hands. How many different hands are possible?
2. The board game *Clue* uses a deck of 21 cards. If 3 people are playing, each person gets 6 cards for their hand. How many different 6-card *Clue* hands are possible?
3. Palmetto Cash 5 is a game offered by the South Carolina Education Lottery. Players choose 5 numbers from the whole numbers between 1 and 38 (inclusive); the player wins the jackpot of $100,000 if the randomizer selects those numbers in any order. How many different sets of winning numbers are possible?

Solution

1. A standard deck has 52 cards, and a hand has 2 cards. Since the order doesn't matter, we use the formula for counting combinations:
$$_{52}C_{2} = \frac{52!}{2!(52-2)!} = \frac{52 \times 51 \times \mathbf{50!}}{2 \times 1 \times \mathbf{50!}} = \frac{52 \times 51}{2} = 1{,}326.$$
2. Again, the order doesn't matter, so the number of combinations is:
$$_{21}C_{6} = \frac{21!}{6!(21-6)!} = \frac{21 \times \mathbf{20} \times 19 \times \mathbf{18} \times 17 \times 16 \times \mathbf{15!}}{\mathbf{6} \times \mathbf{5} \times \mathbf{4} \times \mathbf{3} \times 2 \times 1 \times \mathbf{15!}} = \frac{21 \times 19 \times 17 \times 16}{2} = 54{,}264.$$
3. There are 38 numbers to choose from, and we must pick 5. Since order doesn't matter, the number of combinations is:
$$_{38}C_{5} = \frac{38!}{5!(38-5)!} = \frac{38 \times 37 \times 36 \times 35 \times 34 \times \mathbf{33!}}{5 \times 4 \times 3 \times 2 \times 1 \times \mathbf{33!}} = 501{,}492.$$

YOUR TURN 7.10

1. At a charity event with 58 people in attendance, 3 raffle winners are chosen. All receive the same prize, so order doesn't matter. How many different groups of 3 winners can be chosen?
2. A sorority with 42 members needs to choose a committee with 4 members, each with equal responsibility. How many committees are possible?

The notation and nomenclature used for the number of combinations is not standard across all sources. You'll sometimes see $\binom{n}{r}$ instead of $_{n}C_{r}$. Sometimes you'll hear that expression read as "n choose r" as shorthand for "the number of combinations of n objects taken r at a time."

PEOPLE IN MATHEMATICS

Early Eastern Mathematicians

Although combinations weren't really studied in Europe until around the 13th century, mathematicians of the Middle and Far East had already been working on them for hundreds of years. The Indian mathematician known as Pingala had described them by the second century BCE; Varāhamihira (fl. sixth century) and Halayudha (fl. 10th century) extended Pingala's work. In the ninth century, a Jain mathematician named Mahāvīra gave the formula for combinations that we use today.

In 10th-century Baghdad, a mathematician named Al-Karaji also knew formulas for combinations; though his work is now lost, it was known to (and repeated by) Persian mathematician Omar Khayyam, whose work survives. Khayyam is probably best remembered as a poet, with his *Rubaiyat* being his most famous work.

Meanwhile, in 11th-century China, Jia Xian also was working with combinations, as was his 13th-century successor Yang Hui.

It is not known whether the discoveries of any of these men were known in the other regions, or if the Indians, Persians, and Chinese all came to their discoveries independently. We do know that mathematical knowledge and sometimes texts did get passed along trade routes, so it can't be ruled out.

EXAMPLE 7.11

Combining Combinations with the Multiplication Rule for Counting

The student government at a university consists of 10 seniors, 8 juniors, 6 sophomores, and 4 first-years.

1. How many ways are there to choose a committee of 8 people from this group?
2. How many ways are to choose a committee of 8 people if the committee must consist of 2 people from each class?

Solution

1. There are 28 people to choose from, and we need 8. So, the number of possible committees is ${}_{28}C_8 = 3{,}108{,}105$.
2. Break the selection of the committee members down into a 4-step process: Choose the seniors, then choose the juniors, then the sophomores, and then the first-years, as shown in the table below:

Class	Number of Ways to Choose Committee Representatives
senior	${}_{10}C_2 = 45$
junior	${}_{8}C_2 = 28$
sophomore	${}_{6}C_2 = 15$
first-year	${}_{4}C_2 = 6$

The Multiplication Rule for Counting tells us that we can get the total number of ways to complete this task by multiplying together the number of ways to do each of the four subtasks. So, there are $45 \times 28 \times 15 \times 6 = 113{,}400$ possible committees with these restrictions.

YOUR TURN 7.11

1. How many ways are there to choose a hand of 6 cards from a standard deck with the constraint that 3 are ♠, 2 are ♡, and 1 is ♣?

Check Your Understanding

11. Suppose you want to count the number of ways that you can arrange the apps on the home screen on your phone. Should you use permutations or combinations?

12. Your little brother is packing up for a family vacation, but there's only room for 3 of his toys. If you want to know how many possible groups of toys he can bring, should you use permutations or combinations?

13. Compute ${}_{12}C_{10}$.

14. Compute ${}_{16}C_3$.

15. You're planning a road trip with some friends. Though you have 6 friends you'd consider bringing along, you only have room for 3 other people in the car. How many different possibilities are there for your road trip squad?

16. You're packing for a trip, for which you need 3 shirts and 3 skirts. If you have 8 shirts and 5 skirts that would work for the trip, how many different ways are there to pack for the trip?

SECTION 7.3 EXERCISES

For the following exercises, decide whether the situation describes a permutation or a combination.

1. You're packing for vacation, and you need to pick 5 shirts.
2. You and your friends are about to play a game, and you need to decide who will have the first turn, second turn, and so on.
3. You are watching your favorite reality show, and you want to know how many possibilities there are for the order of finish for the top three.
4. You are going to be working in groups of 4 with your classmates, and you want to know how many possibilities there are for the composition of your group.

For the following exercises, express your answers as whole numbers.

5. ${}_5C_3$
6. ${}_8C_2$
7. ${}_8C_6$
8. ${}_{12}C_3$
9. ${}_{12}C_5$
10. ${}_{14}C_3$
11. ${}_{14}C_{10}$
12. ${}_{15}C_5$
13. ${}_{15}C_{13}$
14. ${}_{18}C_3$
15. ${}_{18}C_6$
16. ${}_{20}C_4$

17. In most variations of the card game poker, a hand consists of 5 cards, where the order doesn't matter. How many different poker hands are there?

18. A professor starts each class by choosing 3 students to present solutions to homework problems to the class. If there are 41 students in the class, in how many different ways can the professor make those selections?

19. An election for at-large members of a school board has 7 candidates; 3 will be elected. How many different ways can those 3 seats be filled?

20. There are 20 contestants on a reality TV show; at the end of the first episode, 10 are eliminated. How many different groups of eliminated contestants are possible?

21. At a horse race, bettors can place a bet called an exacta box. For this bet, the player chooses 2 horses; if those horses finish first and second (in either order), the player wins. In a race with 12 horses in the field, how many possible exacta box bets are there?

The following exercises are about the card game euchre, which uses a partial standard deck of cards: it only has the cards with ranks 9, 10, J, Q, K, and A (for a total of 24 cards). Some variations of the game use the 8s or the 7s and 8s, but we'll stick with the 24-card version.

22. A euchre hand contains 5 cards. How many ways are there to receive a 5-card hand (where the order in which the cards are received doesn't matter, i.e., 9♡, J♡, K♣, 9♠, 10♠ is the same as 9♠ J♡, 9♡, K♣, 10♠)?
23. After all 4 players get their hands, the remaining 4 cards are placed face down in the center of the table. How many different groups of 4 cards are there from this deck?
24. Euchre is played with partners. How many ways are there for 2 partners to receive 5-card hands (where, as above, the order doesn't matter)? Hint: After the first person gets their cards, there are $52 - 5 = 47$ cards left for the second person.

You and 5 of your friends are at an amusement park, and are about to ride a roller coaster. The cars have room for 6 people arranged in 3 rows of 2, so you and your friends will perfectly fill one car.

25. How many ways are there to choose the 2 people in the front row?
26. Assuming the front row has been selected, how many ways are there to choose the 2 people in the middle row?

27. Assuming the first 2 rows have been selected, how many ways are there to choose the 2 people in the back row?
28. Using the Multiplication Rule for Counting and your answers to the earlier parts of this exercise, how many ways are there for your friends to sort yourselves into rows to board the roller coaster?

The University Combinatorics Club has 18 members. Four of them will be selected to form a committee.

29. How many different committees of 4 are possible, assuming all of the duties are shared equally?
30. Instead of sharing responsibility equally, one person will be chosen to be the committee chair. How many different committees are possible? Count these by selecting a chair first, then selecting the remaining 3 members of the committee from the remaining club members and use the Multiplication Rule for Counting. Show your work.
31. Let's count the number of committees with chairs a different way: First, choose 4 people for the committee (as in the first question), then choose 1 of the 4 to be chair. Show your work. Do you get the same number?

Powerball® is a multistate lottery game, which costs $2 to play. Players fill out a ticket by choosing 5 numbers between 1 and 69 (these are the white balls) and then a single number between 1 and 26 (this is the Powerball).

32. How many different ways are there to choose the white balls? Players who match these 5 numbers exactly (but not the Powerball) win $1 million.
33. How many ways are there to choose the Powerball? Players who correctly pick the Powerball win $4.
34. How many ways are there to play the game altogether? Players who match all 5 white balls and the Powerball win (or share) the grand prize. (The grand prize starts at $40 million; if no players win the grand prize, the value goes up for the next drawing. The highest value it has ever reached is $1.586 *billion*!) How many ways are there to fill out a single Powerball ticket?

35. You are in charge of programming for a music festival. The festival has a main stage, a secondary stage, and several smaller stages. There are 40 bands confirmed for the festival. Five of those will play the main stage, and 8 will play the secondary stage. How many ways are there for you to allocate bands to these 2 stages?

7.4 Tree Diagrams, Tables, and Outcomes

Figure 7.10 In genetics, the characteristics of an offspring organism depends on the characteristics of its parents. (credit: "Pea Plant" by Maria Keays/Flickr, CC BY 2.0))

Learning Objectives

After completing this section, you should be able to:

1. Determine the sample space of single stage experiment.
2. Use tables to list possible outcomes of a multistage experiment.
3. Use tree diagrams to list possible outcomes of a multistage experiment.

In the 19th century, an Augustinian friar and scientist named Gregor Mendel used his observations of pea plants to set out his theory of genetic propagation. In his work, he looked at the offspring that resulted from breeding plants with different characteristics together. For applications like this, it is often insufficient to only know in how many ways a