7.1 The Multiplication Rule for Counting

Figure 7.2 The Multiplication Rule for Counting allows us to compute more complicated probabilities, like drawing two aces from a deck. (credit: "Pair of Aces – Poker" by Poker Photos/Flickr, CC BY 2.0)

Learning Objectives

After completing this section, you should be able to:

1. Apply the Multiplication Rule for Counting to solve problems.

One of the first bits of mathematical knowledge children learn is how to count objects by pointing to them in turn and saying: "one, two, three, ..." That's a useful skill, but when the number of things that we need to count grows large, that method becomes onerous (or, for *very* large numbers, impossible for humans to accomplish in a typical human lifespan). So, mathematicians have developed short cuts to counting big numbers. These techniques fall under the mathematical discipline of **combinatorics**, which is devoted to counting.

Multiplication as a Combinatorial Short Cut

One of the first combinatorial short cuts to counting students learn in school has to do with areas of rectangles. If we have a set of objects to be counted that can be physically arranged into a rectangular shape, then we can use multiplication to do the counting for us. Consider this set of objects (Figure 7.3):

Figure 7.3

Certainly we can count them by pointing and running through the numbers, but it's more efficient to group them (Figure 7.4).

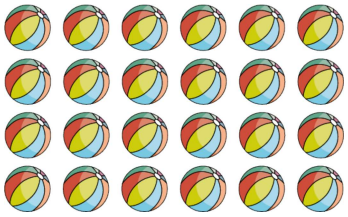

Figure 7.4

If we group the balls by 4s, we see that we have 6 groups (or, we can see this arrangement as 4 groups of 6 balls). Since multiplication is repeated addition (i.e., $6 \times 4 = 4 + 4 + 4 + 4 + 4 + 4$), we can use this grouping to quickly see that there are 24 balls.

Let's generalize this idea a little bit. Let's say that we're visiting a bakery that offers customized cupcakes. For the cake, we have three choices: vanilla, chocolate, and strawberry. Each cupcake can be topped with one of four types of frosting: vanilla, chocolate, lemon, and strawberry. How many different cupcake combinations are possible? We can think of laying out all the possibilities in a grid, with cake choices defining the rows and frosting choices defining the columns

(Figure 7.5).

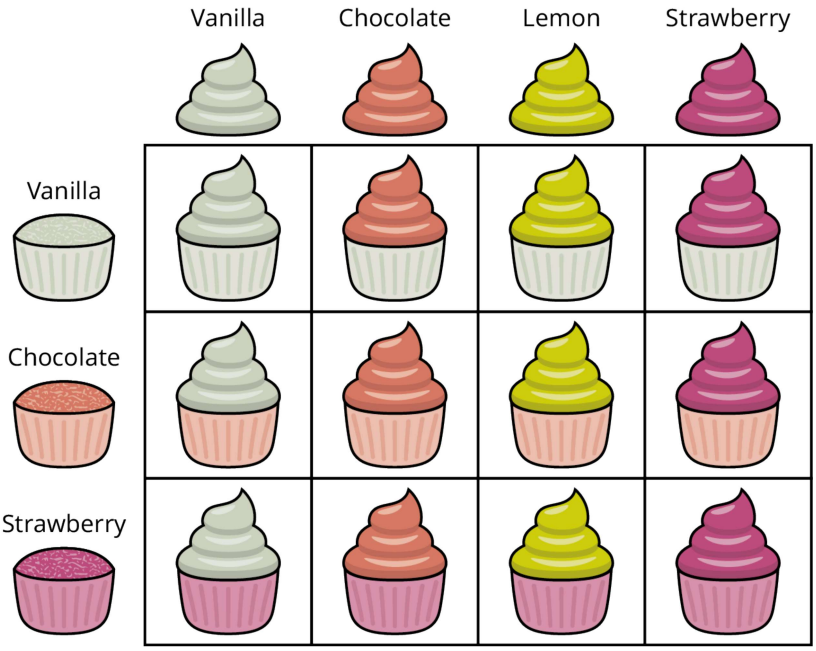

Figure 7.5

Since there are 3 rows (cakes) and 4 columns (frostings), we have $3 \times 4 = 12$ possible combinations. This is the reasoning behind the **Multiplication Rule for Counting**, which is also known as the Fundamental Counting Principle. This rule says that if there are n ways to accomplish one task and m ways to accomplish a second task, then there are $n \times m$ ways to accomplish both tasks. We can tack on additional tasks by multiplying the number of ways to accomplish those tasks to our previous product.

EXAMPLE 7.1

Using the Multiplication Rule for Counting

Every card in a standard deck of cards has two identifying characteristics: a suit (clubs, diamonds, hearts, or spades; these are indicated by these symbols, respectively: ♣, ◇, ♡, ♠) and a rank (ace, 2, 3, 4, 5, 6, 7, 8, 9, 10, jack, queen, and king; the letters A, J, Q, and K are used to represent the words). Each possible pair of suit and rank appears exactly once in the deck. How many cards are in the standard deck?

Solution

Since there are 4 suits and 13 ranks, the number of cards must be $4 \times 13 = 52$ (Figure 7.6).

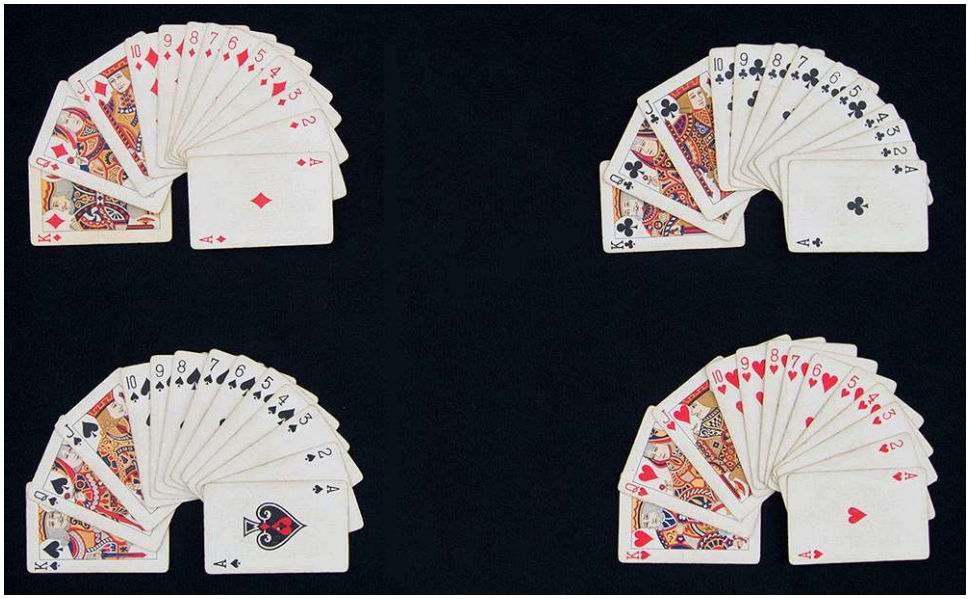

Figure 7.6 Standard Deck of Cards, Sorted by Rank and Suit (credit: "Playing Cards, USS Arkansas" by Naval History & Heritage Command/Flickr, CC BY 2.0)

YOUR TURN 7.1

1. Joe's Pizza Shack offers pizzas with 4 different types of crust and a choice of 15 toppings. How many different one-topping pizzas can be made at Joe's?

EXAMPLE 7.2

Using the Multiplication Rule for Counting for 4 Groups

The University Combinatorics Club has 31 members: 8 seniors, 7 juniors, 5 sophomores, and 11 first-years. How many possible 4-person committees can be formed by selecting 1 member from each class?

Solution

Since we have 8 choices for the senior, 7 choices for the junior, 5 for the sophomore, and 11 for the first-year, there are $8 \times 7 \times 5 \times 11 = 3{,}080$ different ways to fill out the committee.

YOUR TURN 7.2

1. The menu for Joe's Pizza Shack offers pizzas with 4 different types of crust and a choice of 15 toppings. Suppose that Joe's also offers a choice of 3 sauces and 2 cheese blends. How many different one-topping pizzas can be made at Joe's now?

EXAMPLE 7.3

Using the Multiplication Rule for Counting for More Groups

The standard license plates for vehicles in a certain state consist of 6 characters: 3 letters followed by 3 digits. There are 26 letters in the alphabet and 10 digits (0 through 9) to choose from. How many license plates can be made using this format?

Solution

Since there are 26 different letters and 10 different digits, the total number of possible license plates is

$$26 \times 26 \times 26 \times 10 \times 10 \times 10 = 17,576,000.$$

YOUR TURN 7.3

1. At a certain college, ID cards are issued to all students, faculty, and staff. These cards have unique ID codes for each person: a letter to indicate the person's status (S for students, F for faculty, and E for staff), followed by 5 digits and finally 3 letters (these letters can be anything). How many different ID codes can be created using this scheme?

Check Your Understanding

1. A website that lets you build custom belts has 18 different buckles and 30 different straps. How many different belts can be made using those materials?
2. A chain of chicken restaurants offers a combo that includes your choice of 3 or 5 chicken strips, along with your choice of side dish. If there are 7 side dishes, how many different ways are there to build this combo meal?
3. When you flip a coin, there are 2 possible outcomes: heads and tails. Let's say you flip a coin 10 times, and after each you write down the result of the flip (H for heads, T for tails). How many different results (strings of 10 characters, where each is either an H or a T) are possible?
4. A T-shirt company allows shoppers to customize their shirts in several ways. There are 5 sizes, 8 shirt colors, 4 designs, and 5 design colors. How many different shirts can be made?
5. Josephine is trying to build her class schedule for next semester. Because of her work schedule, she has only 4 class periods that can work for her, and she must take 4 classes. If there are 15 classes that she could take during the first period, 18 during the second, 12 during the third, and 8 during the fourth, how many different schedules could Josephine build?

SECTION 7.1 EXERCISES

An ice-cream parlor sells 26 different flavors of ice cream. A basic sundae has one scoop of any flavor of ice cream, your choice of one of 3 sauces, and any one of 8 different toppings.

1. How many different basic sundaes are possible?
2. The ice-cream parlor also sells a medium sundae. The options are the same except it starts with 2 scoops of ice cream, which can be the same flavor or different flavors. How many different medium sundaes are there?
3. The ice-cream parlor also sells a large sundae. The choice of a large sundae allows you to choose any 3 scoops of ice cream, any 2 sauces (they can be the same, or you can choose 2 different ones), and any 3 toppings (that might be 3 servings of the same topping, or 2 servings of one topping and a single serving of another, or 3 different toppings). How many different large sundaes are possible?

4. A company that builds custom computers offers 4 hard drive sizes, 4 memory sizes, 3 graphics cards, and 3 display options. How many computer configurations do they offer, if customers choose one of each customization?
5. A video game allows users to customize their avatars. There are 12 hair styles that users may choose from, as well as 5 hair colors, 8 skin tones, 24 shirts, 12 pants, and 8 shoes. How many different avatars are possible?
6. A small company has 3 divisions: Sales, Research and Development, and Manufacturing. One person from each division will be chosen to create an advisory board for the management group. If there are 8 people in Sales, 15 in Research and Development, and 48 in Manufacturing, how many different compositions of the advisory board are possible?
7. A multiple-choice quiz has 5 questions, each of which has 4 possible answers. How many different ways are there to respond to this quiz?
8. The teacher decides to make the quiz from above a little harder by offering 5 responses on each of the 5 questions. How many ways are there to respond to this quiz?
9. In the United States, radio and television broadcast stations are assigned unique identifiers known as call signs. Call signs consist of 4 letters. The first is either K or W (generally speaking, stations with a K call sign are west of

the Mississippi River and stations with a W call sign are east of the river, though there are several exceptions to this rule); the remaining 3 letters can be anything. How many possible call signs are there under this system?

10. Little sister has asked big brother to play a new game she's invented. It uses a modified deck of cards with 3 suits and only the numbered cards (those with rank 2 through 10). How many cards are in her deck?

11. The board game *Mastermind* has 2 players. One of them is designated the codemaker who creates a code that consists of a series of 4 colors (indicated in the game with 4 colored pegs), which may contain repeats. The other player, who is the codebreaker, tries to guess the code. If there are 6 colors that the codemaker can use to make the code, how many possible codes can be made?

7.2 Permutations

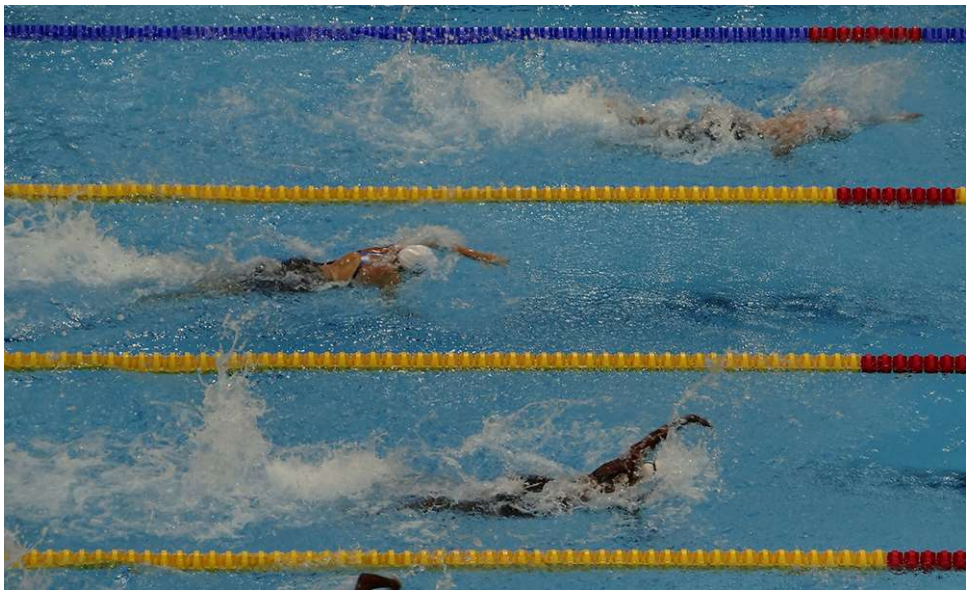

Figure 7.7 We can use permutations to calculate the number of different orders of finish in an Olympic swimming heat. (credit: "London 2012 Olympics Park Stratford London" by Gary Bembridge/Flickr, CC BY 2.0)

Learning Objectives

After completing this section, you should be able to:

1. Use the Multiplication Rule for Counting to determine the number of permutations.
2. Compute expressions containing factorials.
3. Compute permutations.
4. Apply permutations to solve problems.

Swimming events are some of the most popular events at the summer Olympic Games. In the finals of each event, 8 swimmers compete at the same time, making for some exciting finishes. How many different orders of finish are possible in these events? In this section, we'll extend the Multiplication Rule for Counting to help answer questions like this one, which relate to **permutations**. A permutation is an ordered list of objects taken from a given population. The length of the list is given, and the list cannot contain any repeated items.

Applying the Multiplication Rule for Counting to Permutations

In the case of the swimming finals, one possible permutation of length 3 would be the list of medal winners (first, second, and third place finishers). A permutation of length 8 would be the full order of finish (first place through eighth place). Let's use the Multiplication Rule for Counting to figure out how many of each of these permutations there are.

EXAMPLE 7.4

Using the Multiplication Rule for Counting to Find the Number of Permutations

The final heat of Olympic swimming events features 8 swimmers (or teams of swimmers).

1. How many different podium placements (first place, second place, and third place) are possible?
2. How many different complete orders of finish (first place through eighth place) are possible?