11.4 Apportionment Methods

Figure 11.16 Schoolchildren depend on apportionment of resources like laptops among schools and classroom. (credit: “Richmond Public Schools” by Virginia Department of Education/Flickr, CC BY 2.0)

Learning Objectives

After completing this section, you should be able to:

1. Describe and interpret the apportionment problem.
2. Apply Hamilton’s Method.
3. Describe and interpret the quota rule.
4. Apply Jefferson’s Method.
5. Apply Adams’s Method.
6. Apply Webster’s Method.
7. Compare and contrast apportionment methods.
8. Identify and contrast flaws in various apportionment methods.

A Closer Look at the Apportionment Problem

In Standard Divisors, Standard Quotas, and the Apportionment Problem we calculated the standard divisor and the standard quotas in various apportionment scenarios. The results of those calculations routinely led to fractions and decimals of units. However, the seats in the House of Representatives, laptops in a classroom, or a variety of other resources, are indivisible, meaning they cannot be divided up into fractional parts. This leaves a decision to be made. For example, if the standard quota for the number of laptops to be distributed to a classroom is 12.44 units, how do we deal with the fractional part of 0.44? It is unclear if the classroom should receive 12 units, 13 units, or some other value. Let’s try traditional rounding to the nearest whole number value.

EXAMPLE 11.31

Installing Emergency Lights

The board of trustees of a college has recently approved the installation of 70 new emergency blue lights in three parking lots. The number of lights in each lot will be proportionate to the size of the parking lot, which isto be measured in acres. The total number of acres is 34; so the standard divisor is $\frac{34}{70} \approx 0.4857$. The standard quota for each lot is listed in the table below. Use this information to answer each question.

Lot	Acres	Lot's Standard Quota
A	15	$15 \div 0.4857 \approx 30.88$ emergency blue lights
B	9	$9 \div 0.4857 \approx 18.53$ emergency blue lights
C	10	$10 \div 0.4857 = 20.59$ emergency blue lights

1. Use traditional rounding to determine the number of lights assigned to each lot.
2. Find the sum of the values from part 1.
3. Does the sum found in part 2 equal the number of lights available?

Solution

1. If traditional rounding is used, there will be 31, 19, and 21 lights distributed to each lot, respectively.
2. The total of these values is 71.
3. No, the total from part 2 is one more than the number of lights available. In other words, one of the parking lots must get 1 fewer light than apportioned.

YOUR TURN 11.31

The science department of a high school has received a grant for 34 laptops. They plan to apportion them among their six classrooms based on each classroom's student capacity. Use the standard quotas in the table below to answer each question.

Room	Room Capacity	Room's Standard Quota
A	30	$30 \div 4.88 \approx 6.15$ laptops
B	25	$25 \div 4.88 \approx 5.12$ laptops
C	28	$28 \div 4.88 \approx 5.74$ laptops
D	32	$32 \div 4.88 \approx 6.56$ laptops
E	24	$24 \div 4.88 \approx 4.92$ laptops
F	27	$27 \div 4.88 \approx 5.53$ laptops

1. Use traditional rounding to determine the number of laptops assigned to each classroom.
2. Find the sum of the values from part 1.
3. Does the sum found in part 2 equal the number of laptops available?

Example 11.31 demonstrates that we cannot successfully apportion indivisible resources by rounding off each standard quota using traditional rounding. This leaves us with a problem. What is a fair way to distribute the fractional parts of the standard quotas? We will refer to this as the **apportionment problem**. Several methods for making this decision will be discussed.

EXAMPLE 11.32

A-10C Thunderbolt II Aircraft

In 2015, the U.S. Air Force had a fleet of approximately 281 A-10C Thunderbolt II aircraft. Suppose that the Air Force administration wanted to distribute 27 aircrafts across six bases based on the number of qualified pilots stationed at those bases. Use the information in the table below to answer each question.

Base	Pilots
(A) Alpha	13
(B) Bravo	12
(C) Charlie	5
(D) Delta	16
(E) Echo	7
(F) Foxtrot	9

1. Identify the states, the seats, and the state population (the basis for the apportionment) in this scenario.
2. Find the standard divisor for the apportionment of the aircraft. Round to four decimal places as needed. Include the units.
3. Find each Air Force base's standard quota for the apportionment of the aircraft. Round to the nearest hundredth as needed. What are the units?
4. How does this example demonstrate the apportionment problem? Will traditional rounding solve the problem?

Solution

1. The states are the bases, the seats are the aircraft, and the state populations are the pilots at a given base.
2. $\text{Standard Divisor} = \frac{\text{Total Population}}{\text{House Size}} = \frac{13+12+5+16+7+9}{27} = \frac{62}{27} \approx 2.2963$ pilots per aircraft.
3. A $\frac{13}{2.2963} \approx 5.66$, B $\frac{12}{2.2963} \approx 5.23$, C $\frac{5}{2.2963} \approx 2.18$, D $\frac{16}{2.2963} \approx 6.97$, E $\frac{7}{2.2963} \approx 3.05$, F $\frac{9}{2.2963} \approx 3.92$. The units are aircraft.
4. This example demonstrates the apportionment problem because it is not possible to send a fractional number of aircraft to an Air Force base. On the other hand, if we use traditional rounding methods to get whole numbers, the results are $5+5+2+7+3+4=26$ aircraft will be apportioned, which is one less than the number of aircraft that were supposed to beapportioned.

YOUR TURN 11.32

1. The reading coach at an elementary school has 13 gift cards to distribute to their three students as a reward for time spent reading. When they calculated the standard quota for each student based on the number of minutes they student had read, the results were: 4.49 gift cards, 4.03 gift cards, and 4.48 gift cards. How does this demonstrate the apportionment problem?

Hamilton's Method of Apportionment

One of the problems encountered when standard quotas are transformed into whole numbers using traditional rounding is that it is possible for the sum of the values to be greater than the number of seats available. A reasonable way to avoid this is to always round down, even when the first decimal place is five or greater. For example, a standard quota of 12.33 and a standard quota of 12.99 would both round down to 12. This is called the **lower quota**.

EXAMPLE 11.33

Lower Quota for Apportionment of Aircraft

The Air Force administration wants to distribute 27 aircrafts across six bases based on the number of qualified pilots stationed at those bases. The standard quotas for each base are listed in the table below. Use this information to answer the questions.

Base	Standard Quota
(A) Alpha	$\frac{13}{2.2963} \approx 5.66$ aircraft
(B) Bravo	$\frac{12}{2.2963} \approx 5.23$ aircraft
(C) Charlie	$\frac{5}{2.2963} \approx 2.18$ aircraft
(D) Delta	$\frac{16}{2.2963} \approx 6.97$ aircraft
(E) Echo	$\frac{7}{2.2963} \approx 3.05$ aircraft
(F) Foxtrot	$\frac{9}{2.2963} \approx 3.92$ aircraft

1. Give the lower quota for each Air Force base.
2. Find the sum of the lower quotas. By how much does this sum fall short of the actual number of aircraft?

Solution

1. Round down. The lower quota for each Air Force base is 5, 5, 2, 6, 3, 3, respectively.
2. The sum is 24. This is 3 fewer than the actual number of aircraft.

YOUR TURN 11.33

The apportionment of 70 new emergency blue lights in three parking lots is based on acreage. The standard quota for each lot is listed in the table below. Use this information to answer each question.

Lot	Acres	Lot's Standard Quota
A	15	$15 \div 0.4857 \approx 30.88$ emergency blue lights
B	9	$9 \div 0.4857 \approx 18.53$ emergency blue lights
C	10	$10 \div 0.4857 = 20.59$ emergency blue lights

1. Give the lower quota for each parking lot.
2. Find the sum of the lower quotas.
3. By how much does this sum fall short of the actual number of emergency lights?

If the standard quotas are all rounded down, their sum will always be less than or equal to the house size. Then, it would only remain to find a fair way to distribute any remaining seats. Alexander Hamilton, who was a general in the American Revolution, author of the Federalist Papers, and the first secretary of the treasury, took this approach to apportionment.

Steps for Hamilton's Method of Apportionment

There are five steps we follow when applying Hamilton's Method of apportionment:

1. Find the standard divisor.
2. Find each state's standard quota.
3. Give each state the state's lower quota (with each state receiving at least 1 seat).
4. Give each remaining seat one at a time to the states with the largest fractional parts of their standard quotas until

no seats remain.
5. Check the solution by confirming that the sum of the modified quotas equals the house size.

VIDEO

[Hamilton Method of Apportionment (https://openstax.org/r/Hamiltons_method)](https://openstax.org/r/Hamiltons_method)

EXAMPLE 11.34

Hawaiian School Districts

Suppose that the Hawaii State Department of Education has a budget for 616 schools and is doing a research study to determine the equitable number of schools to have in each of the five counties based on the residents under 19 years old, This data is provided in the table below. Using the Hamilton method, calculate how many schools would be funded in each state.

	Hawaii	Honolulu	Kalawao	Kauai	Maui	Total
Residents under age 19	46,310	224,230	20	16,560	38,450	325,570

Solution

Step 1: Calculate the standard divisor. Divide the total population, 325,570, by the house size, 616 seats. The standard divisor is 528.52.

Step 2: Find each state's standard quota:

	Hawaii	Honolulu	Kalawao	Kauai	Maui	Total
Standard Quota	$\frac{46{,}310}{528.52} \approx 87.62$	$\frac{224{,}230}{528.52} \approx 424.26$	$\frac{20}{528.52} \approx 0.04$	$\frac{16{,}560}{528.52} \approx 31.33$	$\frac{38{,}450}{528.52} \approx 72.75$	616

Step 3: Find each state's lower quota and their sum:

	Hawaii	Honolulu	Kalawao	Kauai	Maui	Total
Lower Quota	87	424	1	31	72	615

Step 4: Compare the sum of the states' lower quotas, 615, to the house size, 616. One seat remains to be apportioned and must be given to the state with the largest fractional part: Maui with 0.75. So, the final Hamilton quotas are as follows: Hawaii 87, Honolulu 424, Kalawao 1, Kauai 31, and Maui 73.

Step 5: Find the total to confirm the sum of the quotas equals the house size, 616. Then $87 + 424 + 1 + 31 + 73 = 616$. The apportionment is complete.

YOUR TURN 11.34

1. In the country of Imaginaria, there will be four states: Fictionville, Pretendstead, Illusionham, and Mythbury. Suppose there will be 35 seats in the legislature of Imaginaria. Use Hamilton's method of apportionment to determine the number of seats in each state based on the populations in the following table.

	Fictionville	Pretendstead	Illusionham	Mythbury	Total
Population	71,000	117,000	211,000	1,194,000	1,593,000

TECH CHECK

Apportionment Calculators

Check out websites such as Ms. Hearn Math (https://openstax.org/r/hamilton-calculator) for a free Hamilton apportionment calculator.

This can be a useful tool to confirm your results!

The Quota Rule

A characteristic of an apportionment that is considered favorable is when the final quota values all either result from rounding down or rounding up from the standard quotas. The value that results from rounding down is called the lower quota, and the value that results from rounding up is called the **upper quota**.

As we explore more methods of apportionment, we will consider whether they satisfy the quota rule. If a scenario exists in which a particular apportionment allocates a value greater than the upper quota or less than the lower quota, then that apportionment violates the quota rule and the apportionment method that was used violates the quota rule.

EXAMPLE 11.35

Which Apportionment Method Satisfies the Quota Rule?

Several apportionment methods have been used to allocate 125 seats to ten states and the results are shown in the table below. Determine which apportionments do not satisfy the quota rule and justify your answer.

	State A	State B	State C	State D	State E	State F	State G
Standard Quota	41.26	16.00	5.77	2.64	7.82	10.47	0.21
Lower Quota	41	16	5	2	7	10	0
Upper Quota	42	17	6	3	8	11	1
Method X	43	16	5	2	7	10	1
Method Y	41	16	6	2	8	10	1
Method Z	42	16	7	3	7	9	1

Solution

Look for states such that the number of seats allocated differs from the lower or upper quota. Method X violates the quota rule because State A receives 43 seats instead of 41 or 42. Method Z violates the quota rule because State C receives 7 seats instead of 5 or 6 and State F receives 9 instead of 10 or 11.

YOUR TURN 11.35

1. Apportionment Method V has been used to allocate 125 seats to ten states as shown in the table below. Determine whether the apportionment satisfies the quota rule and justify your answer.

	State A	State B	State C	State D	State E	State F	State G
Standard Quota	41.26	10.70	16.00	13.11	17.00	5.77	2.64
Lower Quota	41	10	16	13	17	5	2
Upper Quota	42	11	17	14	18	6	3
Method V	42	11	17	13	18	4	2

It is possible for an apportionment method to satisfy the quota rule in some scenarios but violate it in others. However, because the Hamilton method always begins with the lower quota and either adds one to it or keeps it the same, the final Hamilton quota will always consist of values that are either lower quota values or upper quota values. When an apportionment method has this characteristic, it is said to satisfy the quota rule. So, we can say:

The Hamilton method of apportionment satisfies the quota rule.

Although the Hamilton method of apportionment satisfies the quota rule, it can result in some unexpected outcomes, which has caused it to pass in and out of favor of the U.S. government over the years. There are several apportionment methods that have been popular alternatives, such as Jefferson's method of apportionment that the founders of Imaginaria should consider.

Jefferson's Method of Apportionment

Another approach to dealing with the fractional parts of the standard quotas is to modify the standard divisor so that the total of the resulting modified lower quotas is the necessary number of seats. This is the approach used by Jefferson.

In Jefferson's method, the change to the standard divisor is made so that the total of the modified lower quotas equals the house size. The change in the standard divisor to get the modified divisor is relatively small. There is not a formula for this. The modified divisor is found by "guess and check." It is important to remember that *increasing* the divisor *decreases* the quotas, but *decreasing* the divisor *increases* the quotas. So, if you need a larger quota, try reducing the divisor, and if you need a smaller quota, try increasing the divisor.

EXAMPLE 11.36

Modifying a Standard Divisor

Suppose the population of a state is 50 and the standard divisor is 12.5.

1. Find the state's standard quota.
2. Increase the standard divisor by 2 units and use the modified divisor to determine the modified quota for the state.
3. Decrease the modified divisor from part 2 by 1.5 units and use the new modified divisor to determine the modified quota for the state.
4. Choose any value of divisor between the value of the modified divisor from part 2 and the value of the modified divisor from part 3 and use it to determine the modified quota for the state.
5. Which modified quota was the largest, the modified quota from part 2, from part 3, or from part 4? Explain why.

Solution

1. The state's standard quota is $\frac{50}{12.5} = 4$.
2. The modified divisor is 14.5. The modified quota is $\frac{50}{14.5} \approx 3.45$.
3. The modified divisor is 13. The modified quota is $\frac{50}{13} \approx 3.85$.
4. One value between 13 and 14.5 is 13.5. With a modified divisor of 13.5, the modified quota is $\frac{50}{13.5} \approx 3.70$.
5. The modified quota from part 3 was the largest because the divisor was the smallest of the three. Dividing the same

number by a smaller value gives a larger result.

YOUR TURN 11.36

Suppose the population of a state is 12 and the standard divisor is 0.225.

1. Find the state's standard quota.
2. Decrease the standard divisor by 0.200 units and use the modified divisor to determine the modified quota for the state.
3. Increase the modified divisor from part 2 by 0.100 units and use the new modified divisor to determine the modified quota for the state.
4. Choose any value of divisor between the value of the modified divisor from part 2 and the value of the modified divisor from part 3 and use it to determine the modified quota for the state.
5. Which modified quota was the smallest, the modified quota from part 2, from part 3, or from part 4? Explain why.

When you use Jefferson's method, you might have to adjust the divisor several times find modified lower quotas that sum to the house size. First, guess what the divisor should be based on the sum of the lower quotas and then increase or decrease it from there based on whether the sum needs to be smaller or larger respectively. If the result still does not produce lower quotas that sum to the house size, adjust again. Keep a record of the values that didn't work to help you narrow your search.

Steps for Jefferson's Method of Apportionment

We take four steps to apply Jefferson's Method of apportionment:

Step 1: Find the standard divisor.

Step 2: Find each state's quota. This will be the standard quota the first time Step 2 is completed and the standard divisor is used, but Step 2 may be repeated as needed using a modified divisor and resulting in modified quotas.

Step 3: Find the states' lower quotas (with each state receiving at least one seat), and their sum.

Step 4: If the sum from Step 3 equals the number of seats, the apportionment is complete. If the sum of the lower quotas is less than the number of seats, reduce the standard divisor. If the sum of the lower quotas is greater than the number of seats, increase the standard divisor. Return to Step 2 using the modified divisor.

EXAMPLE 11.37

Hawaiian State Representative Districts

Suppose that the Hawaii State Department of Education has a budget for 616 schools and is doing a research study to determine the equitable number of schools to have in each of five counties based on the residents under the age of 19. With the data in the table below, apply Jefferson's method to apportion the schools to the counties.

	Hawaii	Honolulu	Kalawao	Kauai	Maui	Total
Residents under Age 19	46,310	224,230	20	16,560	38,450	325,570

Solution

Step 1: The process for finding the standard divisor, standard quotas, and lower quotas is the same in the Hamilton and Jefferson methods of apportionment. We walked through the Hamilton Method in Example 11.34, and following these steps resulted in lower quotas as shown in the table below.

	Hawaii	Honolulu	Kalawao	Kauai	Maui	Total
Standard Quota	$\frac{46,310}{528.52} \approx 87.62$	$\frac{224,230}{528.52} \approx 424.26$	$\frac{20}{528.52} \approx 0.04$	$\frac{16,560}{528.52} \approx 31.33$	$\frac{38,450}{528.52} \approx 72.75$	616
Lower Quota	87	424	1	31	72	615

Step 2: Compare the sum of the states' lower quotas, 615, to the house size, 616. Since 615 is less than 616, use a modified divisor that is less than the standard divisor of 528.52. Try 526.00.

Step 3: Find each state's modified quota, lower quota, and the sum of the lower quotas based on the modified divisor of 526:

	Hawaii	Honolulu	Kalawao	Kauai	Maui	Total
Modified Quota	$\frac{46,310}{526.00} \approx 88.04$	$\frac{224,230}{526.00} \approx 426.29$	$\frac{20}{526.00} \approx 0.04$	$\frac{16,560}{526.00} \approx 31.48$	$\frac{38,450}{526.00} \approx 72.75$	616
Lower Quota	88	426	1	31	72	618

Step 4: The new sum of the lower quotas is 2 units greater than 616. We have overshot the goal. So, increase the divisor to a value between 526.00 and 528.52. Try 527.00.

Step 5: Repeat the process of finding the quotas. Find each state's modified quota, lower quota, and the sum of the lower quotas based on the modified divisor of 526.00:

	Hawaii	Honolulu	Kalawao	Kauai	Maui	Total
Modified Quota	$\frac{46,310}{527.00} \approx 87.87$	$\frac{224,230}{527.00} \approx 425.48$	$\frac{20}{527.00} \approx 0.04$	$\frac{16,560}{527.00} \approx 31.42$	$\frac{38,450}{527.00} \approx 72.96$	616
Lower Quota	87	425	1	31	72	616

Step 6: The new sum of the lower quotas equals the house size. The apportionment is complete.

The apportionment is: Hawaii County 87, Honolulu County 425, Kalawao County 1, Kauai 31, and Maui 72 schools.

When using Jefferson's method, the modified divisors you use may be different from what another person chooses, but final apportionment values will be the same.

YOUR TURN 11.37

1. Let's return to the Imaginarian states of Fictionville, Pretendstead, Illusionham, and Mythbury. Suppose that there are going to be 35 seats in the legislature. This time use Jefferson's method of apportionment to determine the number of seats in each state based on the populations in the table below. How many seats would each state receive?

	Fictionville	Pretendstead	Illusionham	Mythbury	Total
Population	71,000	117,000	211,000	1,194,000	1,593,000

Notice that, in this apportionment, Mythbury received more than the upper quota. Since this apportionment of representatives to Imaginarian states by Jefferson's method does not satisfy the quota rule, we say that:

Jefferson's method violates the quota rule.

We have discussed two apportionment methods: one that satisfies the quota rule and one that does not. Before you decide which method to use in Imaginaria, there are a couple more options to consider.

VIDEO

Jefferson Apportionment Method (https://openstax.org/r/Jeffersons_method)

TECH CHECK

Apportionment Calculators

It is possible to create Excel spreadsheets that complete the calculations necessary to complete a Jefferson Apportionment. In some cases, this work has already been done and posted online. Check out websites such Ms. Hearn Math (https://openstax.org/r/jefferson-calculator) for a free Jefferson apportionment calculator.

This can be a useful tool to confirm your results!

Adams's Method of Apportionment

Adams's method of apportionment is another method of apportionment that is based on a modified divisor. However, instead of basing the changes on the sum of the lower quotas, as Jefferson did, Adams used the upper quotas.

To apply Adams's Method of apportionment, there are four steps we follow:

1. Find the standard divisor.
2. Find each state's quota. This will be the standard quota the first time Step 2 is completed, and the standard divisor is used, but Step 2 may be repeated as needed using a modified divisor and resultingin modified quotas.
3. Find the states' upper quotas and their sum.
4. If the sum from Step 3 equals the number of seats, the apportionment is complete. If the sum of the upper quotas is less than the number of seats, reduce the standard divisor. If the sum of the upper quotas is greater than the number of seats, increase the standard divisor. Return to Step 2 using the modified divisor.

EXAMPLE 11.38

Hawaiian School Districts

As in earlier examples, suppose that the Hawaii State Department of Education has a budget for 616 schools and is doing a research study to determine the equitable number of schools to have in each of the five counties based on the residents under the age of 19. Use the data in the following table and the Adams method to apportion the schools to the counties.

	Hawaii	Honolulu	Kalawao	Kauai	Maui	Total
Residents under Age 19	46,310	224,230	20	16,560	38,450	325,570

Solution

Step 1: The steps of finding the standard divisor and each state's quota are the same in the Jefferson and Adams methods. As in Example 11.37, the standard divisor is 528.52.

Step 2: Find each state's upper quota and their sum:

	Hawaii	Honolulu	Kalawao	Kauai	Maui	Total
Standard Quota	$\frac{46{,}310}{528.52} \approx 87.62$	$\frac{224{,}230}{528.52} \approx 424.26$	$\frac{20}{528.52} \approx 0.04$	$\frac{16{,}560}{528.52} \approx 31.33$	$\frac{38{,}450}{528.52} \approx 72.75$	616
Upper Quota	88	425	1	32	73	619

Step 3: Compare the sum of the states' upper quotas, 619, to the house size, 616. Since 619 is greater than 616, we need to reduce the size of the quotas. Use a modified divisor that is greater than the standard divisor of 528.52. Try 534.00.

Step 4: Find each state's modified quota, upper quota, and the sum of the upper quotas based on the modified divisor of 534:

	Hawaii	Honolulu	Kalawao	Kauai	Maui	Total
Modified Quota	$\frac{46{,}310}{534.00} \approx 86.72$	$\frac{224{,}230}{534.00} \approx 419.91$	$\frac{20}{534.00} \approx 0.04$	$\frac{16{,}560}{534.00} \approx 31.01$	$\frac{38{,}450}{534.00} \approx 72.00$	616
Upper Quota	88	420	1	32	72	613

Step 5: The new sum of the upper quotas is 3 units less than 616. Larger quotas are needed. So, decrease the divisor to a value between 534.00 and 528.52. Try 532.00.

Step 6: Find each state's modified quota, upper quota, and the sum of the upper quotas based on the modified divisor of 532.00:

County	Hawaii	Honolulu	Kalawao	Kauai	Maui	Total
Modified Quota	$\frac{46{,}310}{532.00} \approx 87.05$	$\frac{224{,}230}{532.00} \approx 421.48$	$\frac{20}{532.00} \approx 0.04$	$\frac{16{,}560}{532.00} \approx 31.13$	$\frac{38{,}450}{532.00} \approx 72.27$	616
Upper Quota	88	422	1	32	73	616

Step 7: The new sum of the upper quotas equals the house size. The apportionment is complete.

The apportionment is Hawaii County 88, Honolulu County 422, Kalawao County 1, Kauai 32, and Maui 73 schools.

When using Adams's method, just as with Jefferson's method, the modified divisors you use may be different from what another person chooses, but final apportionment values will be the same.

YOUR TURN 11.38

1. There are four states in Imaginaria: Fictionville, Pretendstead, Illusionham, and Mythbury. Assume there will be 35 seats in the legislature of Imaginaria. Use Adams's method of apportionment to determine the number of seats in each state based on the populations in the table below. How many seats would each state receive?

	Fictionville	Pretendstead	Illusionham	Mythbury	Total
Population	71,000	117,000	211,000	1,194,000	1,593,000

In this apportionment, Mythbury received less than the state's lower quota. So, this apportionment is an example of a scenario in which the Adams's method violates the quota rule.

Adams's method of apportionment violates the quota rule.

So far, only Hamilton's method satisfies the quota rule, but there is one more apportionment method you should consider for Imaginaria.

VIDEO

Adams Method Apportionment Calculator (https://openstax.org/r/Adams_method)

TECH CHECK

Apportionment Calculators

Check out websites such as Ms. Hearn Math (https://openstax.org/r/adams-calculator) for a free Adams Method apportionment calculator.

This can be a useful tool to confirm your results!

Webster's Method of Apportionment

Webster's method of apportionment is another method of apportionment that is based on a modified divisor. However, instead of basing the changes on the sum of the lower quotas, as Jefferson did or the sum of the upper quotas as Adams did, Webster used traditional rounding.

To apply Webster's method of apportionment, there are four steps we take:

1. Find the standard divisor.
2. Find each state's quota. This will be the standard quota the first time Step 2 is completed, and the standard divisor is used, but Step 2 may be repeated as needed using a modified divisor and resulting in modified quotas.
3. Round each state's quota to the nearest whole number and find the sum of these values.
4. If the sum of the rounded quotas equals the number of seats, the apportionment is complete. If the sum of the rounded quotas is less than the number of seats, reduce the divisor. If the sum of the rounded quotas is greater than the number of seats, increase the divisor. Return to Step 2 using the modified divisor.

When using Webster's method, just as with Jefferson's method, the modified divisors you use may be different from what another person chooses, but final apportionment values will be the same.

EXAMPLE 11.39

Hawaiian School Districts

Use the data in the table below to apportion 616 schools to Hawaiian counties. This time, use Webster's method.

	Hawaii	Honolulu	Kalawao	Kauai	Maui	Total
Residents under Age 19	46,310	224,230	20	16,560	38,450	325,570

Solution

To apply Webster's method of apportionment, there are four steps we take:

Step 1: The processes of finding the standard divisor and standard quota are the same in the Jefferson, Adams, and Webster's methods. As in the previous examples, the standard divisor is 528.52.

Step 2: Find each state's rounded quota and their sum:

	Hawaii	Honolulu	Kalawao	Kauai	Maui	Total
Standard Quota	$\frac{46{,}310}{528.52} \approx 87.62$	$\frac{224{,}230}{528.52} \approx 424.26$	$\frac{20}{528.52} \approx 0.04$	$\frac{16{,}560}{528.52} \approx 31.33$	$\frac{38{,}450}{528.52} \approx 72.75$	616
Rounded Quota	88	424	1	31	73	617

Step 3: Compare the sum of the states' rounded quotas, 617, to the house size, 616. Since 617 is greater than 616, we need to reduce the size of the quotas. Use a modified divisor that is greater than the standard divisor of 528.52. Try 534.00.

Step 4: Find each state's modified quota, rounded quota, and the sum of the rounded quotas based on the modified divisor of 534:

	Hawaii	Honolulu	Kalawao	Kauai	Maui	Total
Modified Quota	$\frac{46{,}310}{534.00} \approx 86.72$	$\frac{224{,}230}{534.00} \approx 419.91$	$\frac{20}{534.00} \approx 0.04$	$\frac{16{,}560}{534.00} \approx 31.01$	$\frac{38{,}450}{534.00} \approx 72.00$	616
Upper Quota	87	420	1	31	72	612

Step 5: The new sum of the rounded quotas is 4 units less than 616. Larger quotas are needed. So, decrease the divisor to a value between 534.00 and 528.52. Try 530.00.

Step 6: Find each state's modified quota, rounded quota, and the sum of the rounded quotas based on the modified divisor of 530.00:

	Hawaii	Honolulu	Kalawao	Kauai	Maui	Total
Modified Quota	$\frac{46{,}310}{530.00} \approx 87.38$	$\frac{224{,}230}{530.00} \approx 423.08$	$\frac{20}{530.00} \approx 0.04$	$\frac{16{,}560}{530.00} \approx 31.25$	$\frac{38{,}450}{530.00} \approx 72.55$	616
Upper Quota	87	423	1	31	73	615

Step 7: The new sum of the rounded quotas is 1 unit less than 616. Larger quotas are needed. So, decrease the divisor to a value between 528.52 and 530.00. Try 529.50.

Step 8: Find each state's modified quota, rounded quota, and the sum of the rounded quotas based on the modified divisor of 529.50:

	Hawaii	Honolulu	Kalawao	Kauai	Maui	Total
Modified Quota	$\frac{46,310}{529.50} \approx 87.46$	$\frac{224,230}{529.50} \approx 423.48$	$\frac{20}{529.50} \approx 0.04$	$\frac{16,560}{529.50} \approx 31.27$	$\frac{38,450}{529.50} \approx 72.62$	616
Upper Quota	87	423	1	31	73	615

Step 9: The new sum is still only 1 unit less than 616. Larger quotas are needed, but not much larger. So, decrease the divisor to a value between 528.52 and 529.50. Try 529.30.

Step 10: Find each state's modified quota, rounded quota, and the sum of the rounded quotas based on the modified divisor of 529.30:

	Hawaii	Honolulu	Kalawao	Kauai	Maui	Total
Modified Quota	$\frac{46,310}{529.30} \approx 87.49$	$\frac{224,230}{529.30} \approx 423.63$	$\frac{20}{529.30} \approx 0.04$	$\frac{16,560}{529.30} \approx 31.29$	$\frac{38,450}{529.30} \approx 72.64$	616
Upper Quota	87	424	1	31	73	616

Step 11: The new sum of the rounded quotas equals the house size. The apportionment is complete.

The apportionment is Hawaii County 87, Honolulu County 424, Kalawao County 1, Kauai 31, and Maui 73 schools.

YOUR TURN 11.39

1. If you use Webster's method to apportion 35 legislative seats to the 4 states of Imaginaria, Fictionville, Pretendstead, Illusionham, and Mythbury, with the populations given in the table below, what is the resulting apportionment?

	Fictionville	Pretendstead	Illusionham	Mythbury	Total
Population	71,000	117,000	211,000	1,194,000	1,593,000

So far, we know that the Hamilton method satisfies the quota rule, while the Jefferson and Adams methods do not. The apportionments in the Example and Your Turn above are both scenarios in which the Webster method satisfies the quota rule. Does it always? We have a little more work to do to find out. However, one thing is clear. Not all apportionment methods have the same results. Before you make such an important decision for Imaginaria, it's important to think about the differences in the apportionments that result from these four methods. How will the differences affect the citizens of Imaginaria?

TECH CHECK

Apportionment Calculators

Check out websites such as Ms. Hearn Math (https://openstax.org/r/webster-calculator) for a free Webster Method apportionment calculator.

This can be a useful tool to confirm your results!

Comparing Apportionment Methods

Recall that the four apportionment methods discussed in this chapter differ in two main ways:

- Whether or not a modified divisor is used
- The type of rounding of the quotas that is used

How might these differences affect Imaginarians? In the next two examples, we will compare the results when different apportionment methods are applied to the same scenario.

EXAMPLE 11.40

Hawaiian School Districts with Different Apportionment Methods

Let's use the results from Example 11.34, Example 11.37, Example 11.38, and Example 11.39 to compare the four apportionment methods we have discussed. The following table summarizes the results of the results of the Hamilton, Jefferson, Adams and Webster methods when applied to the apportionment of 616 schools to Hawaiian counties.

	Hawaii	Honolulu	Kalawao	Kauai	Maui
Under 19 years old	46,310	224,230	20	16,560	38,450
Hamilton	87	424	1	31	73
Jefferson	87	425	1	31	72
Adams	88	422	1	32	73
Webster	87	424	1	31	73

1. Do any of the apportionment methods result in the same apportionment? If so, which ones?
2. Which apportionment method would the citizens of the largest county likely favor most and least? Justify your answer.
3. As a group, which apportionment method would the citizens of the other four counties likely favor most and least? Justify your answer.

Solution

1. Yes, the Hamilton and Webster methods result in the same apportionment.
2. The largest county is Honolulu. The citizens would likely favor the Jefferson method of apportionment most since they received the most seats by that method. They would likely favor the Adams method of apportionment least because they received the least number of seats by that method.
3. As a group, the other four counties received 192 seats by either the Hamilton or Webster method, 194 seats by the Adams method, and 191 seats by the Jefferson method. They would likely favor the Adams method the most and favor the Jefferson methods the least.

YOUR TURN 11.40

In Your Turn 11.34, 11.37, 11.38, and 11.39, you apportioned 35 legislative seats among the four states of Imaginaria using the Hamilton, Jefferson, Adams, and Webster methods of apportionment. To understand how the differences in the apportionments might affect Imaginarians, answer these questions.

1. Which apportionment method would the citizens of the largest state likely favor most and least? Justify your answer.
2. As a group, which apportionment method would the citizens of the other three states likely favor most and

least? Justify your answer.

The Adams method favored the smaller states and the Jefferson method favored the larger states in the previous example, but is this the case in general?

Since the Jefferson method begins with the lower quotas, any adjustment to the quotas will be an increase. As you have seen, this is accomplished by using a modified divisor that is smaller than the standard divisor. The next example compares the impact of a decreasing divisor on the modified quotas of large states to the impact of the same size decrease on small states.

EXAMPLE 11.41

Effect of Decreasing Divisors on Modified Quotas

The following table displays the effect of reducing the size of the divisor. Observe the effect this has on the modified quotas of smaller states versus larger states and use the table answer each question.

		Modified Quotas		
State	**Population**	**Divisor: 10,500**	**Divisor: 10,000**	**Divisor: 9,500**
A	10,000	0.95	1	1.05
B	100,000	9.52	10	10.53
C	1,000,000	95.24	100	105.26

1. When the divisor decreases from 10,500 to 10,000, how many representatives are gained by each state based on the lower quota?
2. When the divisor decreases from 10,000 to 9,500, how many representatives are gained by each state based on the lower quota?
3. Which state gains the most representatives each time the divisor is decreased?

Solution

1. Since a state must have at least one seat, State A begins with 1 seat and still has one seat. State B begins with 9 seats and increases to 10 seats. State C begins with 95 seats and increases to 100 seats. So, State A gains 0, B gains 1, and C gains 5 seats.
2. State A begins with 1 and still has 1. State B begins with 10 and still has 10. State C begins with 100 and increases to 105. So, State A gains 0, State B gains 0, and State C gains 5.
3. State C, the largest state, gains the most representatives each time the divisor is decreased.

This example demonstrates that the Jefferson method is biased toward states with larger populations because the modified divisor is smaller than the standard divisor. On the other hand, the Adams's method, which begins with the upper quotas, must increase the standard divisor in order to reduce the quotas. Once again, the effect on the number of seats is greater for the larger states, but this time they are decreased. This means that the Adams's method favors states with smaller populations.

YOUR TURN 11.41

The following table displays the effect of increasing the size of the divisor. Observe the effect this has on the modified quotas of smaller states versus larger states and use the table to answer each question.

		Modified Quotas		
State	**Population**	**Divisor: 11,500**	**Divisor: 12,000**	**Divisor: 12,500**
A	10,000	0.87	0.83	0.8
B	100,000	8.7	8.33	8
C	1,000,000	86.96	83.33	80

1. When the divisor increases from 11,500 to 12,000, how many representatives are lost by each state based on the upper quota?
2. When the divisor increases from 12,000 to 12,500, how many representatives are lost by each state based on the upper quota?
3. Which state loses the most representatives each time the divisor is increased?

Flaws in Apportionment Methods

As we have seen, different apportionment methods can have the same results in some scenarios but different results in others. Citizens of states which receive fewer seats with a particular apportionment method will view the apportionment method as flawed and argue in favor of a different method. This inevitably creates debates regarding the use of one method over another. Methods that favor larger states are likely to be challenged by smaller states, methods that favor smaller states are likely to be challenged by larger states, and methods that violate the quota rule are likely to be challenged by states of any size depending on the circumstances.

Suppose that the State of Hawaii House of Representatives had 51 representatives, each with their own district. Imagine that redistricting were underway, and the representative districts were to be apportioned to each of five counties based on population. The following table shows the apportionment that would result from the use of the Jefferson, Adams, and Webster methods of apportionment.

	Hawaii	Honolulu	Kalawao	Kauai	Maui
Population	201,500	974,600	100	72,300	167,400
Lower Quota	7	35	0	2	6
Upper Quota	8	36	1	3	7
Jefferson	7	35	1	2	6
Adams	7	34	1	3	6
Webster	7	34	1	3	6

From the table, you can see that Hawaii, Kalawao, and Maui receive the same number of seats regardless of the method used. However, citizens of Honolulu would likely reject the Adams and Webster methods arguing that they violate the quota rule. Similarly, citizens of Kauai would probably reject the Jefferson method based on the argument that it unfairly favors the larger states. This scenario demonstrates that the Adams and Webster methods violate the quota rule, but the Jefferson method also violates the quota rule at times. **The Hamilton method is the only method that satisfies the quota rule in all scenarios.** It also consistently favors neither larger nor smaller states. Unfortunately, it can have some strange and results in certain circumstances, which you will see in the next section.

WHO KNEW?

Gerrymandering: A Subtle Way to Impact Apportionment

In addition to your choice of voting method and your choice of apportionment method, there is another important decision to make which could potentially have a huge impact on the fairness of elections in Imaginaria—the creation of electoral districts. In example above, we imagined that there were 51 state legislators in Hawaii, each representing their own district. But how did the legislators decide on the boundaries for these districts? Typically, boundaries are drawn so that each district has approximately the same number of residents, but the percentage of residents in each district with a particular political affiliation can swing the power from one group to another. When the districts are drawn to impact the power of a political party, ethnic or racial group, or other group, this is called gerrymandering. For example, districts can be drawn so that a particular group is spread thinly across districts, increasing the likelihood that they will not have strong representation.

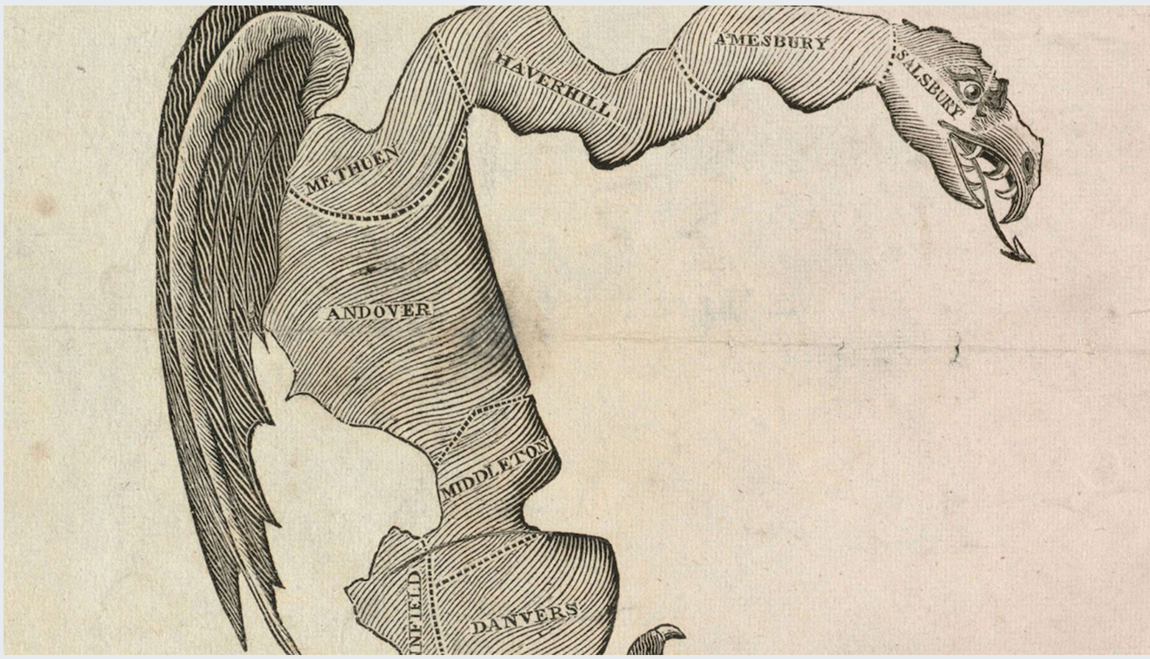

Figure 11.17 This cartoon map conveys the idea that the drawing of the map may impact election outcomes. (credit: “The Gerry Mander”/Wikimedia Commons, Public Domain)

"The Gerry-mander" first appeared in this cartoon-map in the Boston Gazette, March 26, 1812, and was soon reproduced in several other Massachusetts newspapers in response to election district changes initiated by governor Eldridge Gerry. Note that while the practice is named after him, Gerry was not the first to employ it.

PEOPLE IN MATHEMATICS

Jonathan Mattingly

Jonathan Mattingly is a mathematician who was featured in a [*Nature* article titled “The Mathematicians Who Want to Save Democracy” (https://openstax.org/r/mattingly)](https://openstax.org/r/mattingly). Mattingly is a mathematician at Duke University in North Carolina and he runs election simulations based on alternate versions of electoral districts in order to analyze the effects of gerrymandering. He has even been asked to testify as an expert witness in court. Mattingly and other mathematicians who are working on the problem will potentially have an impact on the redistricting that will occur as a result of the 2020 census. (Carrie Arnold, “The Mathematicians Who Want to Save Democracy,” *Nature 546*, 200–202, 2017.)

Check Your Understanding

25. Which of the four apportionment methods discussed in this section does not use a modified divisor?

26. Which of the four apportionment methods discussed in this section satisfies the quota rule?

27. Which of the four apportionment methods discussed in this section is biased toward states with larger populations?

28. Which of the four apportionment methods discussed in this section is biased toward states with smaller populations?

29. Which of the four apportionment methods discussed in this section begin the apportionment with a state's upper quota and adjust down?

30. Which of the four apportionment methods discussed in this section begin the apportionment with a state's lower quota and adjust up?

31. Which of the four apportionment methods discussed in this section use traditional rounding?

32. Does the change from a standard divisor to a modified divisor tend to change the number of seats for larger or smaller states more?

SECTION 11.4 EXERCISES

For the following exercises, use the standard quotas given in the table below.

	State A	State B	State C	State D	State E	State F	Total Seats
Scenario X	17.63	26.62	10.81	16.01	13.69	15.24	100
Scenario Y	12.37	7.59	71.71	6.75	5.76	20.81	125
Scenario Z	3.53	31.56	2.95	5.12	9.84	NA	53

1. Round the standard quota for each state in Scenario X using traditional rounding. Find the sum of the modified quotas. What is the difference between the sum and the house size?
2. Round the standard quota for each state in Scenario Y using traditional rounding. Find the sum of the modified quotas. What is the difference between the sum and the house size?
3. Round the standard quota for each state in Scenario Z using traditional rounding. Find the sum of the modified quotas. What is the difference between the sum and the house size?
4. Find the lower quota for each state in Scenario Y. If each state is allocated its lower quota, how many seats remain to be allocated?
5. Find the lower quota for each state in Scenario X. If each state is allocated its lower quota, how many seats remain to be allocated?
6. Find the lower quota for each state in Scenario Z. If each state is allocated its lower quota, how many seats remain to be allocated?
7. Find the upper quota for each state in Scenario X and determine how much the sum of the upper quotas exceeds the house size.
8. Find the upper quota for each state in Scenario Y and determine how much the sum of the upper quotas exceeds the house size.
9. Find the upper quota for each state in Scenario Z and determine how much the sum of the upper quotas exceeds the house size.
10. Determine the Hamilton apportionment for Scenario Y.
11. Determine the Hamilton apportionment for Scenario X.
12. Determine the Hamilton apportionment for Scenario Z.

For the following exercises, use the information in the table below, which gives standard and final quotas for Methods X, Y, and Z.

	State A	State B	State C	State D	State E
Standard Quota	1.67	3.33	5.00	6.67	8.33
Method X	2	2	5	7	9
Method Y	1	3	5	7	9
Method Z	1	3	5	6	10

13. Does the apportionment resulting from method X satisfy the quota rule? Why or why not?
14. Does the apportionment resulting from method Z satisfy the quota rule? Why or why not?
15. Does the apportionment resulting from method Y satisfy the quota rule? Why or why not?

In the movie *Black Panther,* the hero lives in the fictional country of Wakanda. Imagine that 111 Vibranium artifacts must be distributed among the fortress cities, or birnin, of Wakanda based on the population of each birnin. Use the population and standard quota information in the table below for the following exercises.

	Birnin Djata (D)	Birnin T'Chaka (T)	Birnin Zana (Z)	Birnin S'Yan (S)	Birnin Bashenga (B)	Birnin Azzaria (A)	Total
Residents	26,000	57,000	27,000	18,000	64,000	45,000	237,000
Standard Quota	12.18	26.70	12.65	8.43	29.98	21.08	111

16. Modify the standard quota for each state using traditional rounding. Find the sum of the modified quotas. What is the difference between the sum and the house size?
17. Find the standard lower quota for each state. If each state is allocated its lower quota, how many seats remain to be allocated?
18. Find the standard upper quota for each state and determine how much the sum of the upper quotas exceeds the house size.
19. Use the Hamilton method to apportion the artifacts.
20. Find the modified lower quota for each state using a modified divisor of 2,000. Is the sum of the modified quotas too high, too low, or equal to the house size?
21. Find the modified lower quota for each state using a modified divisor of 2,100. Is the sum of the modified quotas too high, too low, or equal to the house size?
22. Use the Jefferson method to apportion the artifacts. Determine whether it is necessary to modify the divisor. If so, indicate the value of the modified divisor.
23. Does the Jefferson method result in an apportionment that satisfies or violates the quota rule in this scenario?
24. Find the modified upper quota for each state using a modified divisor of 2,250. Is the sum of the modified quotas too high, too low, or equal to the house size?
25. Find the modified upper quota for each state using a modified divisor of 2,150. Is the sum of the modified quotas too high, too low, or equal to the house size?
26. Use the Adams method to apportion the artifacts. Determine whether it is necessary to modify the divisor. If so, indicate the value of the modified divisor.
27. Does the Adams method result in an apportionment that satisfies or violates the quota rule in this scenario?
28. Which method of apportionment, Jefferson or Adams, is a resident of Birnin T'Chaka likely to prefer? Justify your answer.
29. Use the Webster method to apportion the artifacts. Determine whether it is necessary to modify the divisor. If so,indicate the value of the modified divisor.
30. Does the Webster method result in an apportionment that satisfies or violates the quota rule in this scenario?
31. Which of the four methods of apportionment from this section (Hamilton, Jefferson, Adams, or Webster) are the residents of Birnin S'Yan likely to prefer? Justify your answer.

Children from five families—the Chorro family, the Eswaran family, the Javernick family, the Lahde family, and the Stolly family—joined a town Easter egg hunt. When they returned with their baskets, they had 827 eggs! They decided to share their eggs amongst the families based on the number of children in each family. Use the population and standard quota information in the table below for the following exercises.

	(C) Chorro	(E) Eswaran	(J) Javernick	(L) Lahde	(S) Stolly	Total
Children	3	2	4	1	5	15
Standard Quota	155.04	103.36	206.72	103.36	258.40	827

32. Modify the standard quota for each state using traditional rounding. Find the sum of the modified quotas. What is the difference between the sum and the house size?
33. Find the standard lower quota for each state. If each state is allocated its lower quota, how many seats remain to be allocated?
34. Find the standard upper quota for each state, and determine how much the sum of the upper quotas exceeds the house size.
35. Use the Hamilton method to apportion the Easter eggs.
36. Find the modified lower quota for each state using a modified divisor of 0.01800. Is the sum of the modified quotas too high, too low, or equal to the house size?
37. Find the modified lower quota for each state using a modified divisor of 0.01810. Is the sum of the modified quotas too high, too low, or equal to the house size?
38. Use the Jefferson method to apportion the Easter eggs. Determine whether it is necessary to modify the divisor. If so, indicate the value of the modified divisor.
39. Does the Jefferson method result in an apportionment that satisfies or violates the quota rule in this scenario?
40. Find the modified upper quota for each state using a modified divisor of 0.0182. Is the sum of the modified quotas too high, too low, or equal to the house size?
41. Find the modified upper quota for each state using a modified divisor of 0.01816. Is the sum of the modified quotas too high, too low, or equal to the house size?
42. Use the Adams method to apportion the Easter eggs. Determine whether it is necessary to modify the divisor. If so, indicate the value of the modified divisor.
43. Does the Adams method result in an apportionment that satisfies or violates the quota rule in this scenario?
44. Use the Webster method to apportion the Easter eggs. Determine whether it is necessary to modify the divisor. If so, indicate the value of the modified divisor.
45. Does the Webster method result in an apportionment that satisfies or violates the quota rule in this scenario?

For the following exercises, use this information: Suppose that the State of Delaware received 2,000 packs of COVID-19 vaccines, with ten doses per pack. These (unopened) packs must be distributed to the three counties based on total population. Use the population information in the table below to determine how many packs of vaccine will be distributed to each county based on the given apportionment method.

	(N) New Castle	(K) Kent	(S) Sussex
Residents	558,753	180,786	234,225

46. Hamilton’s Method
47. Jefferson’s Method
48. Adams's Method
49. Webster’s Method
50. Notice that the apportionments found in questions 46, 47, 48, and 49 all satisfy the quota rule. Does this contradict the statement, “The Jefferson, Adams, and Webster methods of apportionment all violate the quota rule”? Why or why not?