# 11.1 Voting Methods

**Figure 11.2** President Barack Obama votes in the 2012 election. (credit: Pete Souza/White House, Public Domain)

## Learning Objectives

After completing this section, you should be able to:

1. Apply plurality voting to determine a winner.
2. Apply runoff voting to determine a winner.
3. Apply ranked-choice voting to determine a winner.
4. Apply Borda count voting to determine a winner.
5. Apply pairwise comparison and Condorcet voting to determine a winner.
6. Apply approval voting to determine a winner.
7. Compare and contrast voting methods to identify flaws.

Today is the day that you begin your quest to collaborate on the constitution of Imaginaria! Let's begin by thinking about the selection of a leader who can serve as president. It seems straightforward; if the majority of citizens prefer a particular candidate, that candidate should win. But not all votes are decided by a simple majority. Why not? What are the options?

## Majority versus Plurality Voting

When an election involves only two options, a simple **majority** is a reasonable way to determine a winner. A majority is a number equaling more than half, or greater than 50 percent of the total.

Let's take a look at the outcomes of U.S. presidential elections to understand more. Table 11.1 displays the results of the 2000 U.S. presidential election. Like most presidential elections, this election involved more than two options. If that is the case, is it possible that no single candidate will receive more than half of the votes cast?

| Candidate (Party Label) | Popular Vote Total |
|---|---|
| Al Gore (Democrat) | 50,999,897 |
| George W. Bush (Republican) | 50,456,002 |
| Ralph Nader (Green) | 2,882,955 |
| Patrick J. Buchanan (Reform/Independent) | 448,895 |

**Table 11.1** Results of the Popular Vote for the 2000 U.S. Presidential Election (source: https://www.fec.gov/introduction-campaign-finance/election-and-voting-information/federal-elections-2000/president2000/)

| Candidate (Party Label) | Popular Vote Total |
|---|---|
| Harry Browne (Libertarian) | 384,431 |
| Howard Phillips (Constitution) | 98,020 |
| Other | 134,900 |
| **Total:** | **105,405,100** |

**Table 11.1** Results of the Popular Vote for the 2000 U.S. Presidential Election (source: https://www.fec.gov/introduction-campaign-finance/election-and-voting-information/federal-elections-2000/president2000/)

**EXAMPLE 11.1**

**Majority of Popular Vote in the 2000 U.S. Presidential Election**
Refer back to Table 11.1. Did any single candidate secure the majority of popular votes?

**Solution**
**Step 1:** Calculate 50 percent of 105,405,100 by multiplying the decimal form of 50 percent, which is 0.50, by 158,394,605:
$0.50\,(105{,}405{,}100) = 52{,}702{,}550$

**Step 2:** Determine the minimum number of votes needed to have a majority. The minimum number of votes required is the lowest counting number that is larger than 50 percent of the votes. To have a majority, an individual candidate must have more than 52,702,550; so, a majority candidate must have 52,702,551 votes or more.

**Step 3:** Compare the number of votes each candidate received to 52,702,551. According to the data in Table 11.1, none of the candidates secured a majority.

**YOUR TURN 11.1**

1. According to the Cook Political Report, a total of 158,394,605 votes were cast in the 2020 U.S. presidential election. Of those, 81,281,502 were cast for Joe Biden, 74,222,593 for Donald Trump, and 2,890,510 for other candidates. Although U.S. presidential elections are not determined by popular vote, did either Joe Biden or Donald Trump secure a majority of the votes?

Unlike in the 2000 U.S. presidential election, a candidate won the majority of votes in the 2020 election (see Table 11.1). It is a common occurrence for no single candidate to receive a majority of the votes in an election with more than two candidates. When this occurs, the candidate with the largest portion of the votes is said to have a **plurality**.

**EXAMPLE 11.2**

**Plurality of Popular Vote in the 2000 U.S. Presidential Election**
Refer again to Table 11.1. In the 2000 U.S. presidential election, which candidate had a plurality of popular votes?

**Solution**
Al Gore secured 50,999,897 votes which was more than any other single candidate. Therefore, he had a plurality of the popular votes.

Your plans for Imaginarian elections will likely include primary elections, or preliminary elections to select candidates for a principal or general election. Table 11.2 displays the results of the 2018 U.S. Senate Republican primary for Maryland.

| Top Four Republican Candidates | Votes | Percentage of Party Votes |
|---|---|---|
| Cambell, Tony | 51,426 | 29.22% |
| Chaffee, Chris | 42,328 | 24.05% |
| Grigorian, Christina J. | 30,756 | 17.48% |
| Graziani, John R. | 15,435 | 8.77% |
| **Total Votes** | 175,981 | 100% |

**Table 11.2** (source: https://ballotpedia.org/United_States_Senate_election_in_Maryland_(June_26,_2018_Republican_primary))

**YOUR TURN 11.2**

1. Consider the results of the 2018 U.S. Senate primary for Maryland. Determine which candidate won the primary for the Republicans (R). Did the candidate win a majority or a plurality of Republican votes?

Consider how election by plurality, not majority, is the most common method of selecting candidates for public office.

**WHO KNEW?**

**The U.S. Electorial College: Winning the Presidency without a Plurality**

In the 2000 U.S. presidential election, Al Gore had a plurality of the popular votes, but he did not win the election. Why? This occurred because the U.S. president and vice president are elected by electors rather than a direct vote by the citizens. The electors are part of the Electoral College, a body of people representing the states. Why was the Electoral College created? The Electoral College was created as a compromise between those authors of the U.S. Constitution who believed Congress should elect the president, and those who believed the citizens should vote directly. The popular vote was not recorded until the presidential election of 1824. Since then, only five presidents have been elected without winning a plurality of the popular vote: John Quincy Adams in 1824, Rutherford B. Hayes in 1876, Grover Cleveland in 1888, George Bush in 2000, and Donald Trump in 2016.

## Runoff Voting

Has your family ever debated what to have for dinner? Suppose your family is deciding on a restaurant and exactly half of you want to have pizza but the other half want hamburgers. How do you decide when the result is a tie? You need a tiebreaker!

Will the new democracy of Imaginaria need tiebreakers? When no candidate satisfies the requirements to win the election, a **runoff election**, or second election, is held to determine a winner.

How would runoff voting work in Imaginaria? There are many types of **runoff voting systems**, which are voting systems that utilize a runoff election when the first round does not result in a winner. The method for implementing a runoff election can vary widely, particularly in the criteria used to determine whether a candidate will be on the ballot in the second election. For example, a **two-round system** is a runoff voting system in which only the top candidates advance to the runoff election. In some two-round systems, only the top two candidates are on the second ballot, or it may be any candidate who secures a certain percentage of the vote will advance. The **Hare Method** is another runoff voting system in which only the candidate(s) with the very least votes are eliminated. This can potentially result in several rounds of runoff elections.

**EXAMPLE 11.3**

**Runoff Election for Condominium Association President**

A condominium association elects a new president every two years by a two-round system of voting. If none of the candidates receive a majority, the association charter states that the top two candidates will be eligible to participate in a runoff election. In a particular year, five residents were nominated. The results of the first round are given in the table below.

| Candidate | Votes in First Round |
|---|---|
| Abou | 18 |
| Baiocchi | 10 |
| Campana | 5 |
| Dali | 11 |
| Eugene | 4 |

1. Is there a winner based on the first round? Why or why not?
2. If there is a winner, who won? If there should be a runoff, who will advance to the second round?

**Solution**

1. A majority of 48 total votes is required to win. Begin by finding 50 percent of 48, which is calculated as follows: $0.50(48) = 24$. A majority is 25 or more. No candidate has a majority, so there is no winner based on the first round.
2. Abou and Dali advance to the second round.

**YOUR TURN 11.3**

The student government bylaws of a particular college require that a new president is elected annually by plurality voting. In the event of a tie, the bylaws require the candidate(s) with the fewest votes to be removed from the ballot and a runoff election to be held with the remaining candidates. This process is repeated until a single candidate receives a plurality and wins the election. The results of two voting rounds are given in the table below.

| Candidate | Votes in First Round | Votes in Second Round |
|---|---|---|
| Ferguson | 158 | 168 |
| Garcia | 103 | 104 |
| Hearn | 157 | 180 |
| Isaac | 123 | 123 |
| Jackman | 58 | Eliminated |
| Kelly | 72 | 74 |
| Lim | 158 | 180 |

**1.** Which two candidates tied in the second round? Does this mean there will be a third election?

2. If there was a winner, who won? If there should be a runoff, who must be removed from the ballot?

## Steps to Determine Winner by Plurality or Majority Election with Runoff

To determine the winner by plurality or when a majority election with runoff occurs, we take these three steps:

**Step 1:** If a majority is required to win the election, determine the number of votes needed to achieve a majority. This is the least whole number greater than 50 percent of the total votes. If a majority is not required, move to Step 2.

**Step 2:** Count the number of votes for each candidate in the current round of voting. If a single candidate has enough votes to win a plurality, or a majority as appropriate, then you are done! Otherwise, eliminate a predetermined number of candidates based on the rules of the election. Elimination conditions may vary. For example, the rules may state that the candidate(s) with the fewest votes will be eliminated (as in the Hare method), or that only the candidates meeting a certain threshold will move on (as in a two-round system). Once the appropriate candidates are eliminated, move on to Step 3.

**Step 3:** Hold a runoff election. If the runoff is simulated using a list of voter's preferences, renumber the preferences to reflect the remaining number of options in such a way that the original order of preference is retained. Then repeat Step 2.

Note: The second and third steps may be repeated as many times as necessary for voting procedures that allow multiple runoffs.

**EXAMPLE 11.4**

### Family Dinner Night

The five members of the Chionilis family—Annette, Rene, Seema, Titus, and Galen—have decided to get takeout for dinner. They are trying to decide on a restaurant. The options are Rainbow China, Dough Boys Pizza, Taco City, or Caribbean Flavor. They will use majority election with runoffs where the restaurant with the fewest votes is eliminated in each round. The preferences of each family member are listed by first initial in the table below. An entry of 1 represents the person's first choice; 2, their second; and so on. For example, Annette's second choice is Dough Boys Pizza.

| Options | A | R | S | T | G |
|---|---|---|---|---|---|
| Rainbow China | 1 | 3 | 3 | 1 | 3 |
| Dough Boys Pizza | 2 | 2 | 1 | 2 | 1 |
| Taco City | 3 | 4 | 2 | 4 | 2 |
| Caribbean Flavor | 4 | 1 | 4 | 3 | 4 |

Use the information in the table to answer the following questions.

1. Which common type of runoff voting method is this?
2. List the results of each round of voting based on this information and determine which restaurant was ultimately chosen.

**Solution**

1. the Hare Method
2. **Step 1:** Determine the number of votes necessary to have a majority. There are five family members, so 50 percent of 5 is $0.50(5) = 2.5$. A majority is three or more votes.
   **Step 2:** Count the number of votes for each restaurant in the first election. In a list of voter preferences, the 1s represent the top choice of each voter, which corresponds to their vote in the first round.
   Results of Round 1:
   - Rainbow China — 2 votes
   - Dough Boys — 2 votes
   - Taco City — 0 votes

- Caribbean Flavor — 1 vote

No restaurant received a majority. Eliminate Taco City, which has the fewest first place votes:

| Options | A | R | S | T | G |
|---|---|---|---|---|---|
| Rainbow China | 1 | 3 | 3 | 1 | 3 |
| Dough Boys Pizza | 2 | 2 | 1 | 2 | 1 |
| Caribbean Flavor | 4 | 1 | 4 | 3 | 4 |

**Step 3:** Hold a runoff election. In other words, hold a second round. Since we have a list of the voters' preferences with the eliminated option removed, we will renumber the preferences as first, second, and third so that we keep the original order of preference. The result is that we will count the second-place vote of any voter whose first choice was eliminated.

| Options | A | R | S | T | G |
|---|---|---|---|---|---|
| Rainbow China | 1 | 3 | 2 | 1 | 2 |
| Dough Boys Pizza | 2 | 2 | 1 | 2 | 1 |
| Caribbean Flavor | 3 | 1 | 3 | 3 | 3 |

**Step 4:** Repeat the process from Step 2. Count the number votes for each restaurant in the first-round election. Since we are using a list of preferences, we need to count the number of 1s received by each restaurant.
Results of Round 2:

- Rainbow China — 2 votes
- Dough Boys — 2 votes
- Caribbean Flavor — 1 vote

No single restaurant has three votes. Eliminate Caribbean Flavor, which has the fewest first place votes.

| OPTIONS | A | R | S | T | G |
|---|---|---|---|---|---|
| Rainbow China | 1 | 3 | 2 | 1 | 2 |
| Dough Boys Pizza | 2 | 2 | 1 | 2 | 1 |

**Step 5:** Repeat the process from Step 3. Hold another runoff election. This will be Round 3. Renumber the voters' preferences as first and second this time.

| Options | A | R | S | T | G |
|---|---|---|---|---|---|
| Rainbow China | 1 | 2 | 2 | 1 | 2 |
| Dough Boys Pizza | 2 | 1 | 1 | 2 | 1 |

**Step 6:** Repeat the process from Step 2 one last time. Count the number of first place votes for each remaining restaurant.
Results of Round 3:

- Rainbow China — 2 votes
- Dough Boys — 3 votes

Determine whether any one choice has a majority. Yes! Dough Boys has three votes, so it is the winner!

**YOUR TURN 11.4**

**1.** There are six members on the board of a Parent Teacher Association (PTA) at a local elementary school: the president (P), the vice president (V), the recording secretary (R), the liaison to the administration (L), the treasurer (T), and the chief fundraiser (C). The board must decide which equipment to purchase for the classrooms with moneys from their annual fundraisers. The preferences of the board members are shown in the table below.

| | P | V | R | L | T | C |
|---|---|---|---|---|---|---|
| Option A | 3 | 4 | 1 | 3 | 4 | 1 |
| Option B | 1 | 3 | 2 | 1 | 3 | 2 |
| Option C | 2 | 2 | 3 | 2 | 1 | 4 |
| Option D | 4 | 1 | 4 | 4 | 2 | 3 |

The board uses plurality method and a runoff in the event of a tie, such that the option(s) with the least votes will be eliminated in each round. Which option will be chosen?

## Ranked-Choice Voting

In Example 11.4 and Your Turn 11.4, you were given a list that ordered each voter's preferences. This ordering is called a **preference ranking**. A ballot in which a voter is required to give an ordering of their preferences is a **ranked ballot**, and any voting system in which a voter uses a ranked ballot is referred to as **ranked voting**.

The vote for the Academy Awards uses a ranked ballot. The table below provides an example of a ranked ballot for the 2020 Academy Award nominees for Best Director.

| Candidate for Best Director | Rank top choice as 1, next choice as 2, and so on. | | | | |
|---|---|---|---|---|---|
| Martin Scorsese, *The Irishman* | 1 | 2 | 3 | 4 | 5 |
| Todd Phillips, *Joker* | 1 | 2 | 3 | 4 | 5 |
| Sam Mendes, *1917* | 1 | 2 | 3 | 4 | 5 |
| Quentin Tarantino, *Once Upon a Time in Hollywood* | 1 | 2 | 3 | 4 | 5 |
| Bong Joon-ho, *Parasite* | 1 | 2 | 3 | 4 | 5 |

As you decide on the voting methods that will be used in your new democracy, budget must be a consideration. You

might consider a particular type of ranked voting called **ranked-choice voting (RCV)**, which simulates a series of runoff elections without the usual time and expense involved when voters must repeatedly return to the polls, like we did in Example 11.4.

The method of ranked-choice voting (RCV), also called **instant runoff voting (IRV)**, is a version of the Hare Method, using *preference ranking* so that, if no single candidate receives a majority, the least popular selections can be eliminated and the results can be recounted, without the need for more elections.

⚠ *Ranked voting can be confused with ranked-choice voting, but ranked voting is a more general category which includes ranked-choice voting and several other voting methods.*

As we explore examples of ranked voting, we will summarize the voters' preference rankings using a table in which the top row shows the number of ballots that ranked the options in the same order. Let's practice interpreting the information in this type of table.

**EXAMPLE 11.5**

**Interpreting the Sample Preference Summary**

Refer to the table below containing voters' preference rankings to answer the following questions.

| Number of Ballots | 100 | 200 | 150 | 75 |
|---|---|---|---|---|
| Option A | 1 | 4 | 3 | 4 |
| Option B | 2 | 3 | 4 | 2 |
| Option C | 4 | 2 | 1 | 1 |
| Option D | 3 | 1 | 2 | 3 |

1. How many voters ranked the options in the following order: Option A in fourth place, Option B in second place, Option C in first place, and Option D in third place?
2. How many ballots in total were collected?
3. How many voters indicated that Option C was their first choice?

**Solution**

1. The column farthest to the right displays this ordering. The top entry in this column is 75; so, there were 75 voters who ranked the options in this way.
2. The sum of the top row gives the total number of ballots collected: $100 + 200 + 150 + 75 = 525$. So, there were 525 ballots collected.
3. In the Option C row, there are two entries of 1 which indicate a first choice for that option. These occur in the last two columns. The sum of the top entries in these columns is $150 + 75 = 225$. So, 225 voters indicated Option C as their first choice.

**YOUR TURN 11.5**

A kindergarten class votes on their favorite colors using a ranked ballot. Use the results in the following table to answer the questions.

| Number of Ballots | 4 | 6 | 4 | 7 |
|---|---|---|---|---|
| Red | 2 | 6 | 2 | 5 |
| Blue | 1 | 2 | 1 | 4 |
| Green | 6 | 5 | 5 | 2 |
| Yellow | 5 | 4 | 6 | 3 |
| Purple | 4 | 3 | 4 | 1 |
| Pink | 3 | 1 | 3 | 6 |

1. How many students voted in total?
2. How many students voted for yellow as their favorite color?
3. How many students in the class indicated that blue was their favorite color while green was their least favorite color?

Now that we've covered how to read a summary of preference rankings, let's practice using the ranked-choice method to determine the winner of an election. Recall that ranked-choice voting is still the Hare Method where the candidate with the very least number of votes is eliminated each round until a majority is attained. The difference here is that the voters have completed a ranked ballot, so they don't have to visit the polls multiple times. Here are the steps for ranked-choice voting.

## Steps to Determine Winner by Ranked-Choice Voting

To determine the winner when ranked-choice voting occurs, we take these three steps:

**Step 1:** Determine the number of votes needed to achieve a majority. This is the least whole number greater than 50 percent of the total votes.

**Step 2:** Count the number of first place votes for each candidate. If a candidate has a majority, that candidate wins the election and we are done! Otherwise, eliminate the candidate(s) with the fewest votes and complete Step 3.

**Step 3:** Reallocate the votes to the remaining candidates, and repeat Step 2.

⚠ *Be methodical to avoid arithmetic errors. Make sure that each time you count the number of first place votes they sum to the number of ballots.*

▶ **VIDEO**

How Does Ranked-Choice Voting Work? (https://openstax.org/r/ranked-choice_voting1)

**EXAMPLE 11.6**

**Most Popular Color in Kindergarten**

Let's review the kindergarten class color preferences again, and this time determine which color would be selected based on these results using the ranked-choice method.

| Number of Ballots | 4 | 6 | 4 | 7 |
|---|---|---|---|---|
| Red | 2 | 6 | 2 | 5 |
| Blue | 1 | 2 | 1 | 4 |
| Green | 6 | 5 | 5 | 2 |
| Yellow | 5 | 4 | 6 | 3 |
| Purple | 4 | 3 | 4 | 1 |
| Pink | 3 | 1 | 3 | 6 |

**Solution**

**Step 1:** Determine whether any candidate received a majority. There were 21 ballots. Fifty percent of 21 is $0.50(21) = 10.5$. A majority is 11.

**Step 2:** Count the number of first place votes for each candidate. If a candidate has a majority, that candidate wins the election. Otherwise, eliminate the candidate(s) with the fewest votes.

- Red: 0
- Blue: $4 + 4 = 8$
- Green: 0
- Yellow: 0
- Purple: 7
- Pink: 6

Notice that $8 + 7 + 6 = 21$, which is the total number of ballots. Confirming this helps to catch any arithmetic or counting errors. No candidate has a majority with 11 or more votes. We must eliminate red, green, and yellow which had the fewest votes with 0 each. The remaining votes that must be counted for Round 2 are given in the table below.

| Number of Ballots | 4 | 6 | 4 | 7 |
|---|---|---|---|---|
| Blue | 1 | 2 | 1 | 4 |
| Purple | 4 | 3 | 4 | 1 |
| Pink | 3 | 1 | 3 | 6 |

**Step 3:** Reallocate the votes to the remaining candidates. We can do this by numbering the choices as 1, 2, and 3 in such a way that the order of preference is retained as seen in the table below:

| Number of Ballots | 4 | 6 | 4 | 7 |
|---|---|---|---|---|
| Blue | 1 | 2 | 1 | 2 |
| Purple | 3 | 3 | 3 | 1 |
| Pink | 2 | 1 | 2 | 3 |

**Step 4:** Repeat the process from Step 2. Count the number of first place votes for each candidate. If a candidate has a majority, that candidate wins the election. Otherwise, eliminate the candidate(s) with the fewest votes.

- Blue: $4 + 4 = 8$
- Purple: 7
- Pink: 6

Confirm that $8 + 7 + 6 = 21$. Great! No candidate has 11 or more votes. We must eliminate pink which had the fewest votes with 6. The remaining votes that must be counted for Round 2 are shown in the table below.

| Number of Ballots | 4 | 6 | 4 | 7 |
|---|---|---|---|---|
| Blue | 1 | 2 | 1 | 2 |
| Purple | 3 | 3 | 3 | 1 |

**Step 5:** Repeat the process from Step 3. Reallocate the votes to the remaining candidates. We can do this by numbering the choices as 1 and 2;

| Number of Ballots | 4 | 6 | 4 | 7 |
|---|---|---|---|---|
| Blue | 1 | 1 | 1 | 2 |
| Purple | 2 | 2 | 2 | 1 |

**Step 6:** Repeat the process from Step 2 one last time. Count the number of first place votes for each candidate. If a candidate has a majority, that candidate wins the election. Otherwise, eliminate the candidate(s) with the fewest votes.

- Blue: $4 + 6 + 4 = 14$
- Purple: 7

Blue has a majority and wins the election!

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**VIDEO**

Determine Winner of Election by Ranked-Choice Method (aka Instant Runoff) (https://openstax.org/r/ranked-choice_voting2)

**YOUR TURN 11.6**

**1.** Suppose that 58 *Star Wars* fans were asked to vote for their favorite *Star Wars* character. They were given a ranked ballot, and the results are shown in the following table. Use ranked-choice voting to determine the winner.

| Number of Ballots | 7 | 6 | 10 | 8 | 4 | 5 | 6 | 7 | 2 | 3 |
|---|---|---|---|---|---|---|---|---|---|---|
| Han Solo | 3 | 5 | 1 | 3 | 2 | 4 | 2 | 2 | 1 | 4 |
| Princess Leia | 2 | 1 | 3 | 6 | 4 | 3 | 3 | 1 | 3 | 2 |
| Luke Skywalker | 6 | 6 | 6 | 1 | 5 | 5 | 5 | 4 | 6 | 1 |
| Chewbacca | 4 | 4 | 5 | 4 | 6 | 1 | 6 | 5 | 4 | 5 |

| Number of Ballots | 7 | 6 | 10 | 8 | 4 | 5 | 6 | 7 | 2 | 3 |
|---|---|---|---|---|---|---|---|---|---|---|
| Yoda | 1 | 2 | 4 | 2 | 3 | 2 | 1 | 3 | 2 | 3 |
| R2-D2 | 5 | 3 | 2 | 5 | 1 | 6 | 4 | 6 | 5 | 6 |

## Borda Count Voting

Ranked-choice voting is one type of ranked voting that simulates multiple runoffs based on ranked ballots. Another type of ranked voting is the **Borda count method**, which uses ranked ballots that award candidates points corresponding to the number of candidates ranked lower on each ballot.

To understand how this works, let's review the favorite colors of our kindergarten class from the table below. Let's focus on the votes represented by the first column of the preference summary.

| Number of Ballots | 4 | 6 | 4 | 7 |
|---|---|---|---|---|
| Red | 2 | 6 | 2 | 5 |
| Blue | 1 | 2 | 1 | 4 |
| Green | 6 | 5 | 5 | 2 |
| Yellow | 5 | 4 | 6 | 3 |
| Purple | 4 | 3 | 4 | 1 |
| Pink | 3 | 1 | 3 | 6 |

Each student had six options. This first column tells us that four students ranked blue as their first choice, red as their second choice, pink as their third choice, purple as their fourth choice, yellow as their fifth choice, and green as their sixth choice. Blue was ranked higher than $6 - 1 = 5$ other colors. For each of the four students who completed their ballot in this way, blue would receive five points. Since there were four ballots with this ordering, blue would receive $5(4) = 20$ points from the first column. To determine the total points for each candidate, we have to find the sum of the points they received in each column.

To determine the winner of a contest using the Borda count method, we must compare total number of points earned by each candidate. The candidate with the most points is the winner. Each row of the preference summary corresponds to a single candidate. To find the number of points received by a particular candidate in the preference summary, or their **Borda score**, we will need to focus on the row in which that candidate appears.

Before we practice determining the winner of a Borda count election, let's examine how to find the Borda score for a single candidate.

**EXAMPLE 11.7**

### Most Popular Color in Kindergarten Revisited

Let's review the ballots from the kindergarten class again, as shown in the table below. This time, let's determine the Borda score received by the color purple.

| Number of Ballots | 4 | 6 | 4 | 7 |
|---|---|---|---|---|
| Red | 2 | 6 | 2 | 5 |
| Blue | 1 | 2 | 1 | 4 |
| Green | 6 | 5 | 5 | 2 |
| Yellow | 5 | 4 | 6 | 3 |
| Purple | 4 | 3 | 4 | 1 |
| Pink | 3 | 1 | 3 | 6 |

**Solution**

**Step 1:** Find the number of points received by the candidate in each column.

Column 1: $4 \times (6 - 4) = 4 \times 2 = 8$

Column 2: $6 \times (6 - 3) = 6 \times 3 = 18$

Column 3: $4 \times (6 - 4) = 4 \times 2 = 8$

Column 4: $7 \times (6 - 1) = 7 \times 5 = 35$

**Step 2:** Find the sum of the points received in each column. This is the total number of points received by this candidate: $8 + 18 + 8 + 35 = 69$

This process can also be combined into one step as shown here.

$$\begin{aligned} &4 \times (6 - 4) + 6 \times (6 - 3) + 4 \times (6 - 4) + 7 \times (6 - 1) \\ &= 4 \times 2 + 6 \times 3 + 4 \times 2 + 7 \times 5 \\ &= 8 + 18 + 8 + 35 \\ &= 69 \end{aligned}$$

Purple received 69 points in this election.

*When calculating a Borda score in one step, be careful to use the correct order of operations. Perform the subtraction inside each pair of parentheses first, then perform each multiplication, and then perform each addition.*

**YOUR TURN 11.7**

1. Consider the color preferences of the kindergarten class once more. Using the Borda count method, determine the total number of points the color blue received.

Now let's determine the winner of an election by comparing the Borda scores for each of the candidates.

**EXAMPLE 11.8**

**Determine the Winner by Two Ranked Voting Methods**

Use the table below, which displays a sample preference summary, to answer the questions that follow.

| Number of Ballots | 95 | 90 | 110 | 115 |
|---|---|---|---|---|
| Option A | 4 | 4 | 1 | 1 |
| Option B | 2 | 2 | 2 | 2 |
| Option C | 3 | 1 | 3 | 4 |
| Option D | 1 | 3 | 4 | 3 |

1. Use the ranked-choice voting method to determine the winner of the election.
2. Use the Borda count method to determine the winner of the election.

**Solution**

1. **Step 1:** Determine the number of votes needed to achieve a majority. The number of ballots is $95 + 90 + 110 + 115 = 410$. Fifty percent of 410 is $0.50(410) = 205$. So, 206 votes or more is a majority.

   **Step 2:** Count the number of first place votes for each candidate.

   - Option A has $110 + 115 = 225$
   - Option B has 0
   - Option C has 90
   - Option D has 95

   Since Option A has a majority, Option A is the winner by the ranked-choice method.

2. The Borda scores would be:
   - Option A: $95(4-4) + 90(4-4) + 110(4-1) + 115(4-1) = 675$
   - Option B: $95(4-2) + 90(4-2) + 110(4-2) + 115(4-2) = 820$
   - Option C: $95(4-3) + 90(4-1) + 110(4-3) + 115(4-4) = 475$
   - Option D: $95(4-1) + 90(4-3) + 110(4-4) + 115(4-3) = 490$

   Since Option B has a Borda score of 820 points, Option B is the winner by the Borda count method.

---

**YOUR TURN 11.8**

Answer the following questions using the table below, which summarizes Imaginarian voter preferences.

| Number of Ballots | 12 | 19 | 27 | 29 | 31 | 21 |
|---|---|---|---|---|---|---|
| Candidate A | 1 | 1 | 2 | 2 | 3 | 3 |
| Candidate B | 2 | 3 | 1 | 3 | 1 | 2 |
| Candidate C | 3 | 2 | 3 | 1 | 2 | 1 |

1. Use the ranked-choice voting method to determine the winner of the election.
2. Use the Borda count method to determine the winner of the election.
3. Compare the results of the two methods. Did the same candidate win? What observations can you make about the results?

The Borda count method may seem too complicated to even consider using for Imaginaria, but each voting method has its own pros and cons. The Borda count method, for example, favors compromise candidates over divisive candidates. A **compromise candidate** is not the first choice of most of the voters, but is more acceptable to the population as a whole

than the other candidates. A **divisive candidate** is simultaneously the first choice of a large portion of the voters and the last choice of another large portion of the voters.

In Example 11.8, Candidate A was ranked first by 225 voters, but was ranked last by 185 voters. No voters ranked Candidate A as second or third. It appears that, although Candidate A had the majority of first place votes, there was a significant minority who strongly disliked them. Candidate A was a divisive candidate. Candidate B, on the other hand, was the second choice of every voter, making Candidate B a good compromise. The Borda count method chose Candidate B, a compromise candidate, that was more acceptable to the population as a whole. This scenario is cited by both opponents and proponents of the Borda count method.

**VIDEO**

Determine Winner of Election by Borda Count Method (https://openstax.org/r/Borda_count_method)

## Pairwise Comparison and Condorcet Voting

We have discussed two kinds of ranked voting methods so far: ranked-choice and Borda count. A third type of ranked voting is the **pairwise comparison method**, in which the candidates receive a point for each candidate they would beat in a one-on-one election and half a point for each candidate they would tie. If one candidate earns more points than the others, then that candidate wins. This method is one of several **Condorcet voting methods**, which are methods in which candidates are ranked and then compared pairwise to each other, a candidate having to beat all others in order to win. These methods vary in the way candidates are scored, and there is not always a clear winner. A candidate who wins each possible pairing is known as a **Condorcet candidate**. These terms are named after the Marquis de Condorcet, a French philosopher and mathematician who preferred the pairwise comparison method to the Hare method and made public arguments in its favor.

**PEOPLE IN MATHEMATICS**

Marquis de Condorcet

Condorcet voting methods are named for the Marquis de Condorcet, a French philosopher and mathematician known for, among other accomplishments, writing "*Sur l'admission des femmes au droit de Cité*" ("On the Admission of Women to the Rights of Citizenship"), in 1789, the first published essay on the political rights of women.

For more details visit this Web site.

If you include a Condorcet voting method in the constitution of Imaginaria, the election supervisors may want to use a pairwise comparison matrix like the one in Figure 11.3. It's a tool used to list the number of wins associated with each pairing of two candidates. Each candidate will receive a point for each win and a half a point for each tie. Each pairing is listed twice, once for the number of wins of a candidate over a particular challenger and once for the number of wins of the challenger over that candidate.

| Opponent / Runner | A | B | C |
|---|---|---|---|
| A wins | -- | A over B | A over C |
| B wins | B over A | -- | B over C |
| C wins | C over A | C over B | -- |

**Figure 11.3** Pairwise Comparison Matrix for Three Candidates

## Steps to Determine a Winner by Pairwise Comparison Method Using a Matrix

To determine the winner when the pairwise comparison method is used, we take these three steps:

**Step 1:** On the matrix, indicate a losing matchup by crossing out a box, ☒, and tie match ups by drawing a slash

through the box, \.

**Step 2:** Award each candidate 1 point for a win, half a point for a tie, and 0 points for a loss.

**Step 3:** Identify the winner, which is the candidate with the most points.

**VIDEO**

Determine Winner of Election by Using the Pairwise Comparison Method (https://openstax.org/r/pairwise_comparison_method)

Before you decide on the pairwise comparison method for Imaginaria, review what's involved in constructing a pairwise comparison matrix from a summary of ranked ballots. Then we can use the matrix to determine the winner of the election. Does the winner using the Borda method still win?

**EXAMPLE 11.9**

**Construct and Use a Pairwise Comparison Matrix**

Consider the summary of ranked ballots shown in the table below. Determine the winner of an election using the pairwise comparison method.

| Number of Ballots | 95 | 90 | 110 | 115 |
|---|---|---|---|---|
| Option A | 4 | 4 | 1 | 1 |
| Option B | 2 | 2 | 2 | 2 |
| Option C | 3 | 1 | 3 | 4 |
| Option D | 1 | 3 | 4 | 3 |

1. Construct a pairwise comparison matrix for the sample summary of ranked ballots in the table above.
2. Use the pairwise comparison method to determine a winner.
3. Recall that in Example 11.8, Candidate A won by the ranked-ballot method, and Candidate B won by the Hare method. Did the same candidate win using the pairwise comparison method?
4. Is the winner a Condorcet candidate?

**Solution**

There are four candidates on the ballots. We will need a row and a column for each candidate in addition to the headings, so we will draw a five by five matrix.

| Opponent / Runner | Option A | Option B | Option C | Option D |
|---|---|---|---|---|
| **Option A wins** | -- | A over B | A over C | A over D |
| **Option B wins** | B over A | -- | B over C | B over D |
| **Option C wins** | C over A | C over B | -- | C over D |
| **Option D wins** | D over A | D over B | D over C | -- |

**Figure 11.4** Pairwise Comparison Matrix for Four Candidates

1. **Step 1:** Refer to Figure 11.4 to determine the values that belong in each cell.

- A over B: A is preferred to B in columns 3 and 4. So, A scores $110 + 115 = 225$ points.
- A over C: A is preferred to C in columns 3 and 4. So, A scores 225 points again.
- A over D: Similarly, A scores 225 points.
- B over A: B is preferred to A in columns 1 and 2. So, B scores $95 + 90 = 185$ points.

**Step 2:** Continuing in this way, we complete the pairwise comparison matrix, as shown in Figure 11.5.

| Runner \ Opponent | Option A | Option B | Option C | Option D |
|---|---|---|---|---|
| **Option A wins** | -- | AB 225 | AC 225 | AD 225 |
| **Option B wins** | BA 185 | -- | BC 320 | BD 315 |
| **Option C wins** | CA 185 | CB 90 | -- | CD 200 |
| **Option D wins** | DA 185 | DB 95 | DC 210 | -- |

**Figure 11.5** Pairwise Comparison Matrix for Four Candidates with Vote Counts

2. **Step 1:** Losing pairings are crossed off with an $\boxed{\times}$. In the event of a tie, we will draw a slash, $\boxed{\backslash}$.
**Step 2:** Determine the number of points for each candidate by analyzing their row of wins. Each win is 1 point, each loss, $\boxed{\times}$, is 0 points, and each tie, $\boxed{\backslash}$, is half a point. Construct an additional column for each candidate's points.

| Runner \ Opponent | Option A | Option B | Option C | Option D | Points |
|---|---|---|---|---|---|
| **Option A wins** | -- | AB 225 | AC 225 | AD 225 | 3 |
| **Option B wins** | ~~BA 185~~ | -- | BC 320 | BD 315 | 2 |
| **Option C wins** | ~~CA 185~~ | ~~CB 90~~ | -- | ~~CD 200~~ | 0 |
| **Option D wins** | ~~DA 185~~ | ~~DB 95~~ | DC 210 | -- | 1 |

**Figure 11.6** Pairwise Comparison Matrix for Four Candidates with Pairwise Winners and Points Column Added

**Step 3:** The winner by the pairwise comparison method is Option A with 3 points.

3. Option A was not the winner by the Hare method.
4. The winner, Option A, is a Condorcet candidate because Option A won each pairwise comparison.

⚠ *Notice that the pairwise vote totals are not used to determine the points. Vote totals are only used to determine a win or a loss. Avoid the common error of adding the values in each row to get the points.*

**YOUR TURN 11.9**

According to *Variety* magazine, there were 8,469 eligible Oscar voters in 2020. To make our matrix easier to work with, we've rounded this number up to 8,700. We'd never do this in a real voting situation. Suppose that the voter preferences for the ballot for Best Director had been as shown in the table below.

| Number of Ballots | 2,400 | 2,100 | 1,900 | 1,200 | 1,100 |
|---|---|---|---|---|---|
| Martin Scorsese, *The Irishman* | 3 | 1 | 4 | 5 | 2 |
| Todd Phillips, *Joker* | 1 | 4 | 5 | 2 | 4 |
| Sam Mendes, *1917* | 5 | 5 | 3 | 4 | 5 |
| Quentin Tarantino, *Once Upon a Time in Hollywood* | 4 | 2 | 1 | 3 | 3 |
| Bong Joon-ho, *Parasite* | 2 | 3 | 2 | 1 | 1 |

Refer to the table to answer each question.

1. Construct a pairwise comparison matrix for the Best Director ballots.
2. Who is the "Best Director" according to the pairwise comparison method?
3. Is the winner a Condorcet candidate?

## Three Key Questions

Before you decide if you want to use the pairwise comparison method for Imaginarian elections, let's consider three questions that might affect your decision.

I. Is there always a winner?
II. If there is a winner, is the winner always a Condorcet candidate?
III. If there is a Condorcet candidate, does that candidate always win?

Let's think about why these questions might be important to you if you chose the pairwise comparison method. First, if no candidate meets the criteria to win an election, you will need a backup plan such as a runoff election. Second, if the winner is not a Condorcet candidate, then there is at least one candidate who beat the winner in a pairwise matchup and the supporters of that candidate might question the validity of the election. Finally, if there is a Condorcet candidate who beat every other candidate in a pairwise matchup, it is reasonable to conclude that it would be unfair for anyone else to win. The rest of the examples in this section should illustrate these key concepts.

**EXAMPLE 11.10**

**Rock, Paper, Scissors by Pairwise Comparison**

Suppose that three people are playing the game Rock, Paper, Scissors. On the count of three, each person shows a hand signal for rock, paper, or scissors. Each hand signal beats another hand signal. The group keeps having a tie because Person A always picks rock, Person B always picks paper which beats rock, and Person C always picks scissors which beats paper, and is beaten by rock! This leads to a disagreement about which choice is best. They decide to use the pairwise comparison method determine the winner. Their preference rankings are given in the following table.

| Voters | A | B | C |
|---|---|---|---|
| Rock (R) | 1 | 3 | 2 |
| Paper (P) | 2 | 1 | 3 |
| Scissors (S) | 3 | 2 | 1 |

**Solution**

Construct the comparison matrix:

| Runner \ Opponent | Rock (R) | Paper (P) | Scissors (S) | Points |
|---|---|---|---|---|
| Rock wins | -- | RP 2 | RS 1 | 1 |
| Paper wins | PR 1 | -- | PS 2 | 1 |
| Scissors wins | SR 2 | SP 1 | -- | 1 |

**Figure 11.7** Rock, Paper, Scissors Pairwise Comparison Matrix

There is a tie! There is no winner.

---

Example 11.10 illustrates the answer to the first key question. The pairwise comparison method does not always result in a winner. For example, much like the game of Rock, Paper, Scissors, it is possible for a cyclic pattern to emerge in which each candidate beats the next until the last candidate who beats the first.

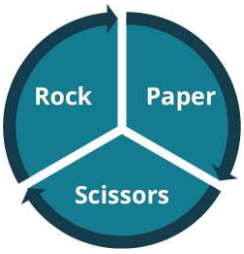


**Figure 11.8** Rock, Paper, Scissors Cyclic Outcome

**YOUR TURN 11.10**

1. A pairwise comparison matrix is given. Determine the winner by the pairwise comparison method. If there is not a winner, explain why. If there is a winner, tell whether the winner is a Condorcet candidate.

| Runner \ Opponent | P | Q | R | S | T | Z |
|---|---|---|---|---|---|---|
| **P wins** | -- | PQ 1 | PR 4 | PS 6 | PT 4 | PZ 5 |
| **Q wins** | QP 5 | -- | QR 6 | QS 4 | QT 1 | QZ 2 |
| **R wins** | RP 2 | RQ 0 | -- | RS 5 | RT 4 | RZ 1 |
| **S wins** | SP 0 | SQ 2 | SR 1 | -- | ST 6 | SZ 5 |
| **T wins** | TP 2 | TQ 5 | TR 2 | TS 0 | -- | TZ 4 |
| **Z wins** | ZP 1 | ZQ 4 | ZR 5 | ZS 1 | ZT 2 | -- |

A Pairwise Comparison Matrix

Now, you have the answer to the second key question. The pairwise comparison matrix in YOUR TURN 11.10 is an example of a scenario where a winner is not a Condorcet candidate.

The answer to the third question is not as clear. If there is a Condorcet candidate, does that candidate win? So far, we have not come across a contradictory example where the Condorcet candidate didn't win, but we cannot know with certainty that it is not possible by looking at examples. Instead, we will need to use some reasoning. Let's review some particular cases of elections with a certain number of candidates, and then we will try to generalize the scenario to an election with $n$ candidates.

**EXAMPLE 11.11**

**Does the Condorcet Candidate Win?**

1. Suppose there is an election with five candidates—A, B, C, D, and E—and that Candidate C is a Condorcet candidate. How many points did Candidate C win?
2. What is the greatest number of points that any one of the other candidates could win?
3. Is it possible for Candidate C to lose or tie?

**Solution**

1. In any pairwise election with five candidates, each candidate must compete against four other candidates. It follows that the most points a single candidate can win is four points, which would occur if the candidate won every matchup. As a Condorcet candidate, Candidate C won all the pairwise matchups against Candidates A, B, D, and E, earning four points.
2. The rest of the candidates lost to Candidate C. The most points a particular candidate could win if they won matchups with each of the other three candidates is three points.
3. Since Candidate C has four points and the rest of the candidates have three points or less, Candidate C is the winner. Therefore, it is not possible for Candidate C to tie or lose.

**YOUR TURN 11.11**

Suppose there is an election with 26 candidates, A through Z, and that Candidate C is a Condorcet candidate.

1. How many points did Candidate C win?
2. What is the greatest number of points that any one of the other candidates could win?
3. Is it possible for Candidate C to lose or tie?

Let's consider a general case where there are $n$ candidates. One of the candidates is a Condorcet candidate. Since the Condorcet candidate wins all matchups, the Condorcet candidate wins $n-1$ points. Since each of the other candidates lost to the Condorcet candidate, the most a single candidate could win is $n-2$. Since the Condorcet candidate won $n-1$ points and each other candidate won $n-2$ points or fewer, the Condorcet candidate is the winner. You have your answer to the third key question! If there is a Condorcet candidate, that candidate is always the winner.

## Approval Voting

The last type of voting system you will consider for your budding democracy is an **approval voting system**. In this system, each voter may approve any number of candidates without rank or preference for one over another (among the approved candidates), and the candidate approved by the most voters wins. This voting system has aspects in common with plurality voting and Condorcet voting methods, but it has characteristics that distinguish it from both. An **approval voting ballot** lists the candidates and provides the option to approve or not approve each candidate.

The term "approval voting" was not used until the 1970s Brams, Steven J.; Fishburn, Peter C. (2007), *Approval Voting*, Springer-Verlag, p. xv, ISBN 978-0-387-49895-9, although its use has been documented as early as the 13th century (Brams, Steven J. (April 1, 2006). *The Normative Turn in Public Choice* (PDF) (Speech). Presidential Address to Public Choice Society. New Orleans, Louisiana.) Approval voting has the appeal of being simpler than ranked voting methods. It also allows an individual voter to support more than one candidate equally. This has appeal for those who do not want a split vote among a few mainstream candidates to lead to the election of a fringe candidate. It also has appeal for those who want an underdog to have a chance of success because voters will not worry about wasting their vote on a candidate who is not believed likely to win.

**EXAMPLE 11.12**

**Rock, Paper, Scissors, Lizard, Spock**

Suppose that Person A/B/C were just about to give up on their game of Rock, Paper, Scissors when they were joined by Person D who reminded them that their updated version, Rock, Paper, Scissors, Lizard, Spock was a far superior game with the added rules that Lizard eats Paper, Paper disproves Spock, Spock vaporizes Rock, Rock crushes Lizard, Lizard poisons Spock, Spock smashes Scissors, and Scissors decapitates Lizard.

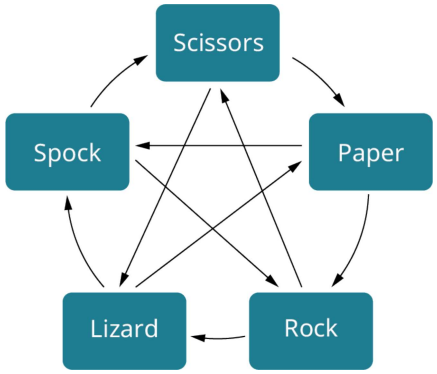


**Figure 11.9** Rock, Paper, Scissors, Lizard, Spock Dominance

Person D encourages their friends to hold a new election. This time, for the sake of simplicity, the group decides to use approval voting to determine the best move in the game. The summary of approval ballots for Rock, Paper, Scissors, Lizard, Spock is given in the table below.

| Voters | A | B | C | D |
|---|---|---|---|---|
| Rock | Yes | No | No | No |
| Paper | No | Yes | No | No |
| Scissors | No | No | Yes | No |

| Voters | A | B | C | D |
|---|---|---|---|---|
| Lizard | No | No | No | Yes |
| Spock | Yes | Yes | Yes | Yes |

**Solution**

Count the number of approval votes for each candidate by counting the number of "Yes" votes in each row of the table.

- Rock: 1
- Paper: 1
- Scissors: 1
- Lizard: 1
- Spock: 4

Spock is the winning candidate, approved by four voters!

**YOUR TURN 11.12**

1. The Chionilis family is trying to decide on a restaurant again, but now they don't want to deal with multiple runoffs or even ranking. They will use the approval voting method shown in the following table. Each family member will approve their top two choices. Rainbow China won when multiple runoffs were used. Find the winner for tonight's dinner.

| Options | A | R | S | T | G | E | M | D |
|---|---|---|---|---|---|---|---|---|
| Rainbow China | Yes | No | No | Yes | No | No | Yes | Yes |
| Dough Boys Pizza | Yes | Yes | Yes | Yes | No | Yes | No | Yes |
| Taco City | No | No | Yes | No | Yes | Yes | No | No |
| Caribbean Flavor | No | Yes | No | No | Yes | No | Yes | No |

**EXAMPLE 11.13**

**The Chionilis Family Is Hungry Again!**

The eight members of the Chionilis family—Annette, Rene, Seema, Titus, Galen, Elena, Max and Demitri—have another decision to make. Approval voting worked out nicely the last time. They are going to use it again, but this time, Annette, Rene, Seema, and Galen are feeling a little indecisive. They can't narrow their choice down to two. They will approve their three top choices, but the other family members will only approve two. These choices are reflected in the following table. Determine the restaurant that will be chosen.

| Options | A | R | S | T | G | E | M | D |
|---|---|---|---|---|---|---|---|---|
| Rainbow China | Yes | Yes | Yes | Yes | Yes | No | Yes | Yes |
| Dough Boys Pizza | Yes | Yes | Yes | Yes | No | Yes | No | Yes |

| Options | A | R | S | T | G | E | M | D |
|---|---|---|---|---|---|---|---|---|
| Taco City | Yes | No | Yes | No | Yes | Yes | No | No |
| Caribbean Flavor | No | Yes | No | No | Yes | No | Yes | No |

**Solution**

Count the number of approval votes for each restaurant by counting the number of "Yes" votes in each row.

- Rainbow China: 7
- Dough Boys Pizza: 6
- Taco City: 4
- Caribbean Flavor: 3

This time, Rainbow China won!

---

**YOUR TURN 11.13**

1. What would be the outcome of the election if every member of the Chionilis family approved their top three choices from the table below?

| Options | A | R | S | T | G |
|---|---|---|---|---|---|
| Rainbow China | 1 | 3 | 3 | 1 | 3 |
| Dough Boys Pizza | 2 | 2 | 1 | 2 | 1 |
| Taco City | 3 | 4 | 2 | 4 | 2 |
| Caribbean Flavor | 4 | 1 | 4 | 3 | 4 |

## Compare and Contrast Voting Methods to Identify Flaws

Wow! We have covered a lot of options for the voting methods. Now, you need to decide which one is best for Imaginaria. Imaginarians might consider characteristics of certain voting systems desirable and others undesirable. In some cases, voters may consider these undesirable traits to be flaws in a voting system that are significant enough to motivate them to reject that system. If you are feeling a bit overwhelmed by this decision, maybe it would help to read about the experiences of others who have faced similar questions.

Consider the 2000 U.S. presidential election in which Green Party candidate Ralph Nader and Reform Party candidate Pat Buchanan were on the ballet running against the mainstream candidates, Democrat Al Gore and Republican George W. Bush. The voting results for Florida are given in Table 11.3.

| Candidate | Party | Votes | Percentage |
|---|---|---|---|
| **(G) George W. Bush** | Republican | 2,912,790 | 48.85% |
| **(A) Al Gore** | Democrat | 2,912,253 | 48.84% |

**Table 11.3** Florida Results in the 2000 U.S. Presidential Election (source: https://www.fec.gov/resources/cms-content/documents/FederalElections2000_PresidentialGeneralElectionResultsbyState.pdf)

| Candidate | Party | Votes | Percentage |
|---|---|---|---|
| **(R) Ralph Nader** | Green | 97,488 | 1.63% |
| **(P) Pat Buchanan** | Reform | 17,484 | 0.29% |
| **(H) Harry Brown** | Libertarian | 16,415 | 0.28% |
| **(O) 7 Other Candidates** | Other | 6,680 | 0.11% |
| | **Total** | 5,963,110 | |

**Table 11.3** Florida Results in the 2000 U.S. Presidential Election (source: https://www.fec.gov/resources/cms-content/documents/FederalElections2000_PresidentialGeneralElectionResultsbyState.pdf)

In more than one state, Buchanan was able to split the Republican vote enough to allow Gore to win that state. Nader split the Democrat vote in Florida and New Hampshire by enough votes to prevent Gore from winning those states. Had Gore won either state, he would have had enough electoral votes to win the election. Instead, Bush won. This is an example of a flaw in the plurality system of voting: the spoiler.

A **spoiler** is a less popular candidate who takes votes from a more popular candidate with similar positions, swinging the race to another candidate with vastly different views that they would not support. This encourages voters not to vote for the candidate that they perceive to be the best, but instead for the candidate they can live with who they perceive to have a better chance of winning. Some voters may prefer a method such as approval voting, which does not have this trait in common with plurality voting.

**EXAMPLE 11.14**

### The Spoiler Controversy

Because the vote counts for George W. Bush and Al Gore differed by only 537 votes, many Democrats blamed Ralph Nader and the Green Party for their loss. Let's consider how the election results might have differed if the approval voting method had been used.

Use Table 11.3 and the following assumptions to extrapolate the results of an approval method election:

- 100 percent of Pat Buchanan supporters would approve George W. Bush.
- 100 percent of Ralph Nader supporters would approve Al Gore.
- 72 percent of Libertarians would approve George W. Bush.
- 28 percent of Libertarians would approve Al Gore (as was roughly the known percentage at the time according to the Cato Institute).
- 50 percent of the supporters of other candidates would approve George Bush while 50 percent would approve Al Gore.

**Solution**

**Step 1:** Create a summary of approval ballots based on the given assumptions. For the Libertarian candidate, 72 percent of 16,415 of the votes is $0.72(16,415) = 11,819$ and 28 percent is $0.28(16,415) = 4,596$. For the other candidates, 50 percent of the votes is $0.50(6,680) = 3,340$.

| Number of Votes | 2,912,790 (G) | 2,912,253 (A) | 97,488 (R) | 17,484 (B) | 11,819 (72% H) | 4,596 (28% H) | 3,340 (50% O) | 3,340 (50% O) |
|---|---|---|---|---|---|---|---|---|
| **(G) George W. Bush** | Yes | No | No | Yes | Yes | No | No | Yes |
| **(A) Al Gore** | No | Yes | Yes | No | No | Yes | Yes | No |

| Number of Votes | 2,912,790 (G) | 2,912,253 (A) | 97,488 (R) | 17,484 (B) | 11,819 (72% H) | 4,596 (28% H) | 3,340 (50% O) | 3,340 (50% O) |
|---|---|---|---|---|---|---|---|---|
| **(R) Ralph Nader** | No | No | Yes | No | No | No | No | No |
| **(P) Pat Buchanan** | No | No | No | Yes | No | No | No | No |
| **(H) Harry Brown** | No | No | No | No | Yes | Yes | No | No |
| **(O) 7 Other Candidates** | No | No | No | No | No | No | Yes | Yes |

**Step 2:** Count the number of approval votes for each candidate.

- George W. Bush: $2,912,790 + 17,484 + 11,819 + 3340 = 2,945,433$
- Al Gore: $2,912,253 + 97,488 + 4,596 + 3,340 = 3,017,677$
- Ralph Nader: $97,488$
- Pat Buchanan: $17,484$
- Harry Brown: $11,819 + 4,596 = 16,415$
- Other Candidates: $6,680$

In this scenario, Al Gore is the winner.

**YOUR TURN 11.14**

1. Extrapolate the results of an approval method election using Table 11.3, the assumptions from Example 11.14, and the additional assumptions that the supporters of Al Gore would all approve Ralph Nader and that half of the supporters of George Bush would approve Pat Buchanan while half would approve Harry Brown.

The results in Example 11.14 and Your Turn 11.14 highlight one of the characteristics of approval voting. Ralph Nader moved up from a distant third place finish to a close second place finish when Al Gore's supporters approved him on their ballots. In this way, fringe candidates have a better chance of winning, which some voters consider a flaw but others consider a benefit.

Another aspect of approval voting systems that is a concern to many voters is that candidates in approval elections might encourage their loyal supporters to approve them and only them to avoid giving support to any other candidate. If this occurred, the election in effect becomes a traditional plurality election. This is a flaw that cannot occur in an instant runoff system since all candidates are ranked.

**EXAMPLE 11.15**

**Three Habitable Planets**

In the future, humans have explored distant solar systems and found three habitable planets which could be colonized. Since it will take all available resources to colonize one planet, humans must agree on the planet. Planet A has the most comfortable climate and most plentiful resources, but it is the farthest from Earth making travel to the planet a challenge. Planet B is half the distance but will require more resources to make comfortable. Planet C is the least suitable of the three and terraforming will be required, but it is close enough to make travel between Earth and Planet C possible on a more regular basis. The table below provides the voter preferences for the colonization of each planet.

| Percentage of Voters | 45% | 15% | 40% |
|---|---|---|---|
| **Planet A** | 1 | 3 | 3 |
| **Planet B** | 2 | 1 | 2 |
| **Planet C** | 3 | 2 | 1 |

If the entire population were able to vote, determine the winning planet using each of the methods listed below.

1. Plurality
2. Ranked-choice method
3. Borda count

**Solution**

1. The plurality method only considers the top choice of each voter. By this system, Planet A has 45 percent of the vote, Planet B has 15 percent of the vote, and Planet C has 40 percent of the vote. Planet A wins.
2. Using either instant runoff or a two-round system, Planet B with only 15 percent of the vote will be eliminated in the first round. In Round 2, the 15 percent that voted for Planet B would vote for their second choice, Planet C. This leaves Planet A with 45 percent and Planet C with 55 percent. Planet C has a majority and wins the election.
3. To find the Borda score for each candidate, imagine there are exactly 100 voters. Then the summary of ranked ballots looks like:

| Out of 100 Voters | 45 | 15 | 40 |
|---|---|---|---|
| Planet A | 1 | 3 | 3 |
| Planet B | 2 | 1 | 2 |
| Planet C | 3 | 2 | 1 |

The Borda score for each candidate is as follows:
Planet A: $(3-1)(45)+(3-3)(15)+(3-3)(40)=90$
Planet B: $(3-2)(45)+(3-1)(15)+(3-2)(40)=115$
Planet C: $(3-3)(45)+(3-2)(15)+(3-1)(40)=95$
Planet B wins.

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**YOUR TURN 11.15**

The juniors at a high school in Central Florida are voting for a theme park to visit for an end of the year field trip. The options are Disney's four theme parks: Animal Kingdom, Magic Kingdom, EPCOT Center, or Hollywood Studios.

| Percentage of Voters | 35% | 25% | 30% |
|---|---|---|---|
| **Animal Kingdom** | 1 | 3 | 4 |
| **Magic Kingdom** | 2 | 1 | 3 |

| Percentage of Voters | 35% | 25% | 30% |
|---|---|---|---|
| Epcot Center | 3 | 4 | 1 |
| Hollywood Studios | 4 | 2 | 2 |

Determine the outcome of the vote by each of the given voting methods.

1. Plurality
2. Ranked-choice method
3. Borda count

The election in Example 11.15 involves a scenario in which there are two extreme candidates, Planet A and Planet B, and a moderate candidate, Planet C. The supporters of the extreme candidates prefer the moderate candidate to the other extremist ones. This makes Planet C a compromise candidate. In this case, both the plurality method and ranked-choice voting resulted in the election of one of the extreme candidates, but the Borda count method elected the compromise candidate in this scenario. Depending on a person's perspective, this may be perceived as a flaw in either ranked-choice and plurality systems, or the Borda count method.

In Fairness in Voting Methods, we will analyze the fairness of each voting system in greater detail using objective measures of fairness.

**TECH CHECK**

Voting Calculators

It is possible to create Excel spreadsheets that complete the calculations necessary to determine the winner of an election by various voting methods. In some cases, this work has already been done and posted online. As you practice applying the various voting methods that could be used in Imaginaria, quick Internet search will lead to sites such as Ms. Hearn Math (https://openstax.org/r/calculators) with free specialty calculators.

These sites can be a great way to check your results!

## Check Your Understanding

1. Name three voting methods that use a ranked ballot.
2. Determine whether the following statement is true or false: The same ranked ballots may result in a different winner depending on which voting method is used.
3. Determine whether the following statement is true or false. A majority candidate is always a Condorcet candidate.
4. The _____ method is a system of voting using ranked ballots in which each candidate is awarded points corresponding to the number of candidates ranked lower on each ballot.
5. The _____ method is a system of voting using ranked ballots (or multiple elections) in which each candidate receives a point for each candidate they would beat in a one-on-one election and half a point for each candidate they would tie.
6. The _____ method is a runoff voting system in which only the candidate(s) with the very least votes are eliminated.
7. Explain the differences between two-round voting and ranked-choice voting.

## SECTION 11.1 EXERCISES

For the following exercises, identify the winning candidate based on the described voter profile, if possible. If it is not possible, state so. Explain your reasoning.

1. In a plurality election, the candidates have the following vote counts: A 125, B 132, C 149, D 112.
2. In the first round of a ranked-choice election with three candidates—A, B, and C. Candidate A received 55 first place rankings; Candidate B received 25; and Candidate C received 30.
3. The pairwise matchup points for each candidate were: A 1, B $1\frac{1}{2}$, D $\frac{1}{2}$.
4. In a Borda count election, the candidates have the following Borda scores: A 15, B 11, C 12, D 16.
5. There is a pairwise comparison election with candidates A, B, and C. Candidate A had the most first choice rankings, Candidate B has the highest Borda score, and Candidate C is a Condorcet candidate.
6. In the first round of a ranked-choice election with three candidates—A, B, and C—Candidate A received 20 first place rankings, Candidate B received 25, and Candidate C received 30.

For the following exercises, use the table.

| O'Malley | De La Fuente | Clinton | Sanders | Other |
|---|---|---|---|---|
| 110,227 | 67,331 | 17,174,432 | 13,245,671 | 322,276 |

Popular Vote in the 2016 U.S. Democratic Presidential Primary (source: Federal Election Commission, *Federal Elections 2016 Report*)

7. Calculate the number of votes required to have a majority of the popular vote in the 2016 U.S. Democratic Presidential Primary.
8. Which candidate had a plurality? Did this candidate have a majority?

For the following exercises, use the given table.

| Candidate | Votes |
|---|---|
| Bush | 281,189 |
| Trump | 13,783,037 |
| Cruz | 7,455,780 |
| Rubio | 3,354,067 |
| Carson | 822,242 |
| Kasich | 4,198,498 |
| Other | 337,714 |

Popular Vote in the 2016 U.S. Republican Presidential Primary (source: Federal Election Commission, *Federal Elections 2016 Report*)

9. Calculate the number of votes required to have a majority of the popular vote in the 2016 U.S. Republican Presidential Primary.
10. Which candidate had a plurality? Did this candidate have a majority?

For the following exercises, use Table 11.4 and Table 11.5.

11. Suppose the Republican Primary in 2016 was a two-round system. Would there be a second round? Why or why not? If so, which candidates would advance to the second round?
12. Suppose the Democratic Primary in 2016 was a two-round system. Would there be a second round? Why or why not? If so, which candidates would advance to the second round?

13. Suppose the Democratic Primary in 2016 used the Hare method. Would there be a second round? Why or why not?
14. Suppose the Republican Primary in 2016 used the Hare method. Would there be a second round? Why or why not?

For the following exercises, use the following table and the Hare method.

| Options | A | B | C | D | E |
|---|---|---|---|---|---|
| Candidate 1 | 1 | 3 | 3 | 1 | 3 |
| Candidate 2 | 2 | 1 | 1 | 2 | 4 |
| Candidate 3 | 3 | 4 | 2 | 4 | 1 |
| Candidate 4 | 4 | 2 | 4 | 3 | 2 |

15. How many votes are needed to win by the Hare method?
16. How many votes does each candidate receive in Round 1?
17. Which candidates advance to Round 2?
18. How many votes does each remaining candidate receive in Round 2?
19. Will there be a third round? Why or why not?
20. Which candidate wins the election?

For the following exercises, use the sample summary of ranked ballots in the given table.

| Number of Ballots | 10 | 20 | 15 | 5 |
|---|---|---|---|---|
| Option A | 1 | 4 | 3 | 4 |
| Option B | 2 | 3 | 4 | 2 |
| Option C | 4 | 2 | 1 | 3 |
| Option D | 3 | 1 | 2 | 1 |

Sample Summary of Ranked Ballots

21. How many votes were recorded, and how many are required to have a majority?
22. How many voters indicated that Option A was their first choice?
23. How many voters indicated that Option B was their first choice?
24. How many voters indicated that Option A was their last choice?
25. How many voters indicated that Option B was their last choice?
26. Use ranked-choice voting to determine the two candidates in the final round and the number of votes they each receive in that round.
27. Is there a winning candidate? If so, which candidate? Justify your answer.

Suppose that 55 Star Wars fans were asked to vote for their favorite new Star Wars character. They were given a ranked ballot, and the results are shown in the table. Use this table and ranked-choice voting for the following exercises.

| Number Of Ballots | 7 | 6 | 10 | 8 | 4 | 5 | 6 | 7 | 2 |
|---|---|---|---|---|---|---|---|---|---|
| **Finn** | 3 | 5 | 1 | 3 | 2 | 4 | 2 | 2 | 1 |
| **Rey** | 2 | 1 | 3 | 2 | 4 | 3 | 3 | 1 | 3 |
| **Poe** | 1 | 6 | 4 | 1 | 5 | 5 | 5 | 4 | 2 |
| **BB8** | 4 | 2 | 2 | 4 | 6 | 1 | 6 | 5 | 4 |
| **Rose** | 6 | 4 | 5 | 6 | 3 | 2 | 1 | 3 | 6 |
| **Kylo** | 5 | 3 | 6 | 5 | 1 | 6 | 4 | 6 | 5 |

Favorite New Heroes in Star Wars Sequels Ballot Preferences

**28**. How many votes does each candidate get on the first round of voting?
**29**. How many votes are required to get a majority?
**30**. Which candidates remain in the final round, and how many votes do they have?
**31**. Who is the winner of the election?

Refer to Table 11.6 for the following exercises.

**32**. Find the Borda score for each candidate.
**33**. Compare your results from question 32 to those from question 26. Compare the winner and the second-place candidate using the Borda count method to those using the ranked-choice method. Are they the same?

Refer to Table 11.7 for the following exercises.

**34**. Find the Borda score for each candidate.
**35**. Compare your results from question 34 to those from question 30. Compare the winner and the second-place candidate using the Borda count method to those using the ranked-choice method. Are they the same?

For the following exercises, use the table below.

| Number of Ballots | 100 | 80 | 110 | 105 | 55 |
|---|---|---|---|---|---|
| Option A | 1 | 1 | 4 | 4 | 2 |
| Option B | 2 | 2 | 2 | 3 | 1 |
| Option C | 4 | 4 | 1 | 1 | 4 |
| Option D | 3 | 3 | 3 | 2 | 3 |

**36**. Do any candidates appear to be divisive candidates? Justify your answer.
**37**. Do any candidates appear to be compromise candidates? Justify your answer.
**38**. How many votes are required for a majority?
**39**. Which candidate is eliminated first by the ranked-choice method?
**40**. Which candidate is eliminated second by the ranked-choice method?
**41**. Which candidate is the winner by the ranked-choice method?
**42**. What are the Borda scores for each candidate?
**43**. Which candidate is the winner by the Borda count method?
**44**. Which method resulted in a win for the compromise candidate: ranked-choice voting or the Borda count method or both?

Use the pairwise comparison matrix in the given figure for the following exercises.

| Opponent / Runner | Q | R | S | T |
|---|---|---|---|---|
| **Q wins** | -- | QR 3 | QS 2 | QT 1 |
| **R wins** | RQ 1 | -- | RS 3 | RT 2 |
| **S wins** | SQ 2 | SR 1 | -- | ST 3 |
| **T wins** | TQ 3 | TR 2 | TS 1 | -- |

Pairwise Comparison Matrix for Candidates Q, R, S and T

**45**. Analyze the pairwise comparison matrix. Display the pairings in a table and indicate the winner of each matchup by marking an $\boxed{\times}$ through the losing matchups and a single slash $\boxed{\backslash}$ through the ties.
**46**. Calculate the points received by each candidate in the pairwise comparison matrix.
**47**. Determine whether there is a winner of the pairwise comparison election represented by the matrix. If there is a winner, determine whether the winner is a Condorcet candidate.

Use the pairwise comparison matrix in the given figure for the following exercises.

| Opponent / Runner | U | V | W | X | Y |
|---|---|---|---|---|---|
| **U wins** | -- | UV 1 | UW 3 | UX 3 | UY 4 |
| **V wins** | VU 5 | -- | VW 6 | VX 4 | VY 1 |
| **W wins** | WU 3 | WV 0 | -- | WX 5 | WY 4 |
| **X wins** | XU 3 | XV 2 | XW 1 | -- | XY 6 |
| **Y wins** | YU 2 | YV 5 | YW 2 | YX 0 | -- |

Pairwise Comparison Matrix for Candidates U, V, W, X, and Y

**48**. Analyze the pairwise comparison matrix. Display the pairings in a table and indicate the winner of each matchup.
**49**. Calculate the points received by each candidate in the pairwise comparison matrix.
**50**. Determine whether there is a winner of the pairwise comparison election represented by the matrix. If there is a winner, determine whether the winner is a Condorcet candidate and explain your reasoning.
**51**. In J.K. Rowling's *Harry Potter* series, Albus Dumbledore was the headmaster of Hogwarts for many years. Imagine that an election is to be held to find his successor. Severus Snape, the head of Slytherin House, will be running against the heads of Gryffindor and Ravenclaw, Minerva McGonagall and Filius Flitwick. Use the preference rankings for each candidate in the following table to construct a pairwise comparison matrix.

| Percentage of Vote | 25% | 40% | 35% |
|---|---|---|---|
| (S) Snape | 1 | 3 | 3 |
| (M) McGonagall | 3 | 1 | 2 |
| (F) Flitwick | 2 | 2 | 1 |

**52**. Analyze the pairwise comparison matrix you constructed for question 51. Display the pairings in a table and indicate the winner of each matchup.
**53**. Use the pairwise comparison matrix from questions 51 and 52 to find the number of points earned by each candidate. Who is the winner by the pairwise comparison method?
**54**. Is the winner of the Hogwarts headmaster election a Condorcet candidate? Explain how you know.
**55**. The women of *The Big Bang Theory* decide to hold their own approval voting election to determine the best option in Rock, Paper, Scissors, Lizard, Spock. Use the summary of their approval ballots in the table below to determine the number of votes for each candidate. Determine the winner, or state that there is none.

| Voters | Penny | Bernadette | Amy |
|---|---|---|---|
| Rock | Yes | No | No |
| Paper | Yes | Yes | No |
| Scissors | Yes | Yes | Yes |
| Lizard | No | No | No |
| Spock | Yes | No | Yes |

For the following exercises, use the table below.

| Percentage of Vote | 40% | 35% | 25% |
|---|---|---|---|
| Candidate A | 1 | 3 | 2 |
| Candidate B | 2 | 1 | 3 |
| Candidate C | 3 | 2 | 1 |

**56**. Which candidate is the winner by the ranked-choice method?
**57**. Suppose that they used the approval method and each voter approved their top two choices. Which candidate is the winner by the approval method?
**58**. Which candidate is the winner by the Borda count method?