10.4 Polygons, Perimeter, and Circumference

Figure 10.62 Geometric patterns are often used in fabrics due to the interest the shapes create. (credit: "Triangles" by Brett Jordan/Flickr, CC BY 2.0)

Learning Objectives

After completing this section, you should be able to:

1. Identify polygons by their sides.
2. Identify polygons by their characteristics.
3. Calculate the perimeter of a polygon.
4. Calculate the sum of the measures of a polygon's interior angles.
5. Calculate the sum of the measures of a polygon's exterior angles.
6. Calculate the circumference of a circle.
7. Solve application problems involving perimeter and circumference.

In our homes, on the road, everywhere we go, polygonal shapes are so common that we cannot count the many uses. Traffic signs, furniture, lighting, clocks, books, computers, phones, and so on, the list is endless. Many applications of polygonal shapes are for practical use, because the shapes chosen are the best for the purpose.

Modern geometric patterns in fabric design have become more popular with time, and they are used for the beauty they lend to the material, the window coverings, the dresses, or the upholstery. This art is not done for any practical reason, but only for the interest these shapes can create, for the pure aesthetics of design.

When designing fabrics, one has to consider the perimeter of the shapes, the triangles, the hexagons, and all polygons used in the pattern, including the circumference of any circular shapes. Additionally, it is the relationship of one object to another and experimenting with different shapes, changing perimeters, or changing angle measurements that we find the best overall design for the intended use of the fabric. In this section, we will explore these properties of polygons, the perimeter, the calculation of interior and exterior angles of polygons, and the circumference of a circle.

Identifying Polygons

A **polygon** is a closed, two-dimensional shape classified by the number of straight-line sides. See Figure 10.63 for some examples. We show only up to eight-sided polygons, but there are many, many more.

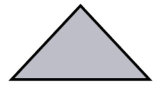
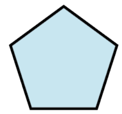
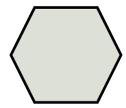
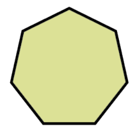
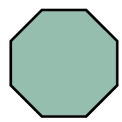
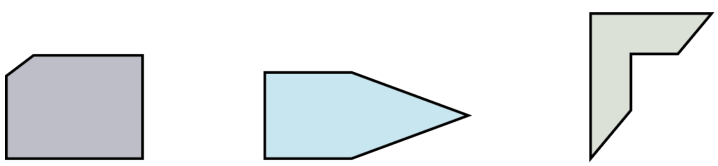
Number of Sides	Name	Shape
3	Triangle	
4	Quadrilateral	
5	Pentagon	
6	Hexagon	
7	Heptagon	
8	Octagon	

Figure 10.63 Types of Polygons

If all the sides of a polygon have equal lengths and all the angles are equal, they are called **regular polygons**. However, any shape with sides that are line segments can classify as a polygon. For example, the first two shapes, shown in Figure 10.64 and Figure 10.64, are both pentagons because they each have five sides and five vertices. The third shape Figure 10.64 is a hexagon because it has six sides and six vertices. We should note here that the hexagon in Figure 10.64 is a concave hexagon, as opposed to the first two shapes, which are convex pentagons. Technically, what makes a polygon concave is having an interior angle that measures greater than $180°$. They are hollowed out, or cave in, so to speak. Convex refers to the opposite effect where the shape is rounded out or pushed out.



(a) (b) (c)

Figure 10.64 Polygons

While there are variations of all polygons, quadrilaterals contain an additional set of figures classified by angles and whether there are one or more pairs of parallel sides. See Figure 10.65.

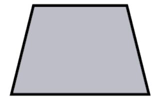
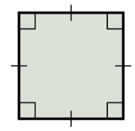
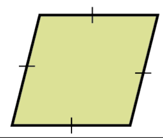
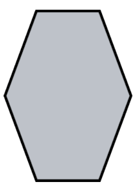
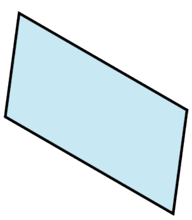
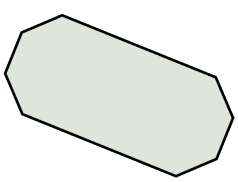
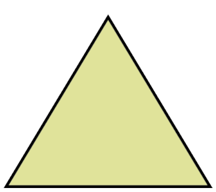
Description	Shape
A **trapezoid** has one pair of parallel sides.	
A **parallelogram** has two sets of parallel sides and no right angles.	
A **rectangle** is a parallelogram with four right angles and two sets of parallel sides.	
A **square** is a rectangle with four equal sides.	
A **rhombus** is a parallelogram with all sides equal, two sets of parallel sides.	

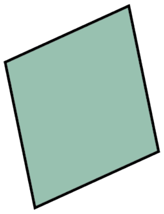
Figure 10.65 Types of Quadrilaterals

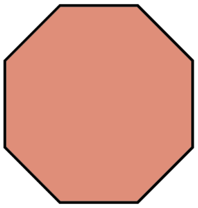
EXAMPLE 10.24

Identifying Polygons

Identify each polygon.

1. 
2. 
3. 
4. 

5.


6.


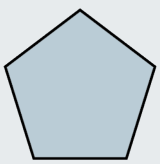
Solution

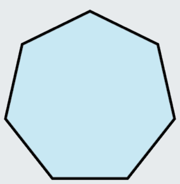
1. This shape has six sides. Therefore, it is a hexagon.
2. This shape has four sides, so it is a quadrilateral. It has two pairs of parallel sides making it a parallelogram.
3. This shape has eight sides making it an octagon.
4. This is an equilateral triangle, as all three sides are equal.
5. This is a rhombus; all four sides are equal.
6. This is a regular octagon, eight sides of equal length and equal angles.

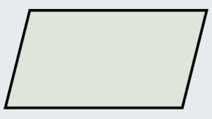
YOUR TURN 10.24

Identify the shape.

1.


2.


3.


4.


EXAMPLE 10.25

Determining Multiple Polygons

What polygons make up Figure 10.66?

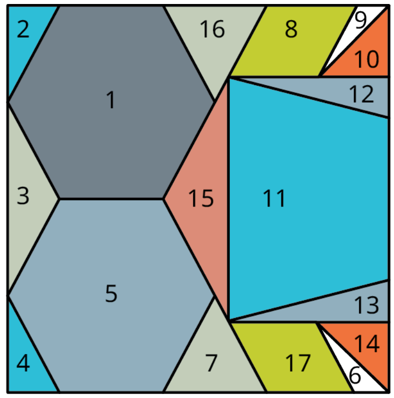


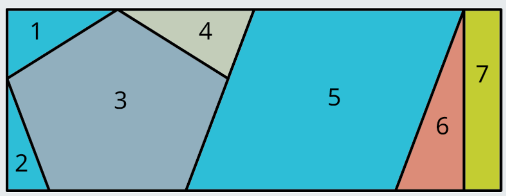
Figure 10.66

Solution

Shapes 1 and 5 are hexagons; shapes 2, 3, 4, 6, 7, 9, 10, 12, 13, 14, 15, and 16 are triangles; shapes 8 and 17 are parallelograms; and shape 11 is a trapezoid.

YOUR TURN 10.25

1. What polygons make up the figure shown?



Perimeter

Perimeter refers to the outside measurements of some area or region given in linear units. For example, to find out how much fencing you would need to enclose your backyard, you will need the perimeter. The general definition of **perimeter** is the sum of the lengths of the sides of an enclosed region. For some geometric shapes, such as rectangles and circles, we have formulas. For other shapes, it is a matter of just adding up the side lengths.

A **rectangle** is defined as part of the group known as quadrilaterals, or shapes with four sides. A rectangle has two sets of parallel sides with four angles. To find the perimeter of a rectangle, we use the following formula:

FORMULA

The formula for the perimeter P of a rectangle is $P = 2L + 2W$, twice the length L plus twice the width W.

For example, to find the length of a rectangle that has a perimeter of 24 inches and a width of 4 inches, we use the formula. Thus,

$$\begin{aligned} 24 &= 2l + 2(4) \\ &= 2l + 8 \\ 24 - 8 &= 2l \\ 16 &= 2l \\ 8 &= l \end{aligned}$$

The length is 8 units.

The perimeter of a regular polygon with n sides is given as $P = n \cdot s$. For example, the perimeter of an equilateral triangle, a triangle with three equal sides, and a side length of 7 cm is $P = 3(7) = 21\text{ cm}$.

EXAMPLE 10.26

Finding the Perimeter of a Pentagon

Find the perimeter of a regular pentagon with a side length of 7 cm (Figure 10.67).

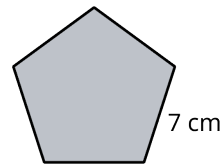


Figure 10.67

Solution

A regular pentagon has five equal sides. Therefore, the perimeter is equal to $P = 5(7) = 35 \text{ cm}$.

YOUR TURN 10.26

1. Find the perimeter of a square table that measures 30 inches across from one side to its opposite side.

EXAMPLE 10.27

Finding the Perimeter of an Octagon

Find the perimeter of a regular octagon with a side length of 14 cm (Figure 10.68).

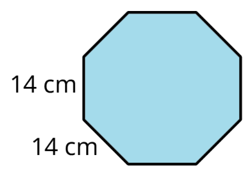


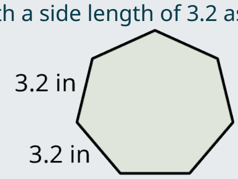
Figure 10.68

Solution

A regular octagon has eight sides of equal length. Therefore, the perimeter of a regular octagon with a side length of 14 cm is $P = 8(14) = 112 \text{ cm}$.

YOUR TURN 10.27

1. Find the perimeter of a regular heptagon with a side length of 3.2 as shown in the figure.



Sum of Interior and Exterior Angles

To find the sum of the measurements of interior angles of a regular polygon, we have the following formula.

FORMULA

The sum of the interior angles of a polygon with n sides is given by

$$S = (n - 2)180^\circ.$$

For example, if we want to find the sum of the interior angles in a parallelogram, we have

$$\begin{aligned} S &= (4-2)180^\circ \\ &= 2(180) = 360^\circ. \end{aligned}$$

Similarly, to find the sum of the interior angles inside a regular heptagon, we have

$$\begin{aligned} S &= (7-2)\,180^\circ \\ &= 5(180) \\ &= 900^\circ. \end{aligned}$$

To find the measure of each interior angle of a regular polygon with n sides, we have the following formula.

FORMULA

The measure of each interior angle of a regular polygon with n sides is given by

$$a = \frac{(n-2)180^\circ}{n}.$$

For example, find the measure of an interior angle of a regular heptagon, as shown in Figure 10.69. We have

$$a = \frac{(7-2)180^\circ}{7} = 128.57^\circ.$$

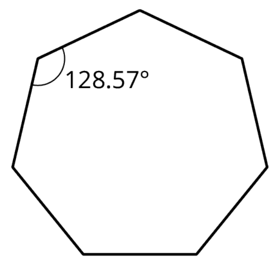


Figure 10.69 Interior Angles

EXAMPLE 10.28

Calculating the Sum of Interior Angles

Find the measure of an interior angle in a regular octagon using the formula, and then find the sum of all the interior angles using the sum formula.

Solution

An octagon has eight sides, so $n = 8$.

Step 1: Using the formula $a = \frac{(n-2)180^\circ}{8}$:

$$\begin{aligned} a &= \frac{(8-2)180^\circ}{8} \\ &= \frac{(6)180^\circ}{8} \\ &= 135^\circ. \end{aligned}$$

So, the measure of each interior angle in a regular octagon is 135°.

Step 2: The sum of the angles inside an octagon, so using the formula:

$$\begin{aligned} S &= (n-2)180^\circ \\ &= (8-2)180^\circ \\ &= 6(180) \\ &= 1{,}080^\circ. \end{aligned}$$

Step 3: We can test this, as we already know the measure of each angle is 135°. Thus, $8(135^\circ) = 1{,}080^\circ$.

YOUR TURN 10.28

1. Find the measure of each interior angle of a regular pentagon and then find the sum of the interior angles.

EXAMPLE 10.29

Calculating Interior Angles

Use algebra to calculate the measure of each interior angle of the five-sided polygon (Figure 10.70).

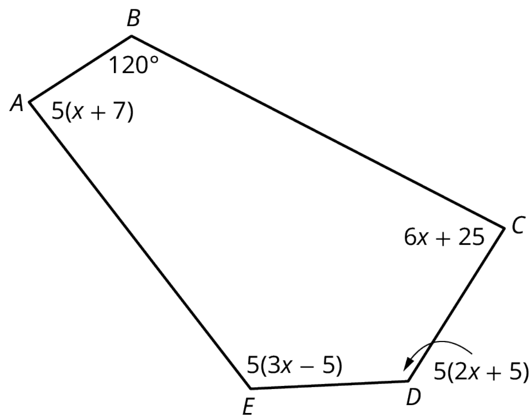


Figure 10.70

Solution

Step 1: Let us find out what the total of the sum of the interior angles should be. Use the formula for the sum of the angles in a polygon with n sides: $S = (n - 2)180^\circ$. So, $S = (5 - 2)180^\circ = 540^\circ$.

Step 2: We add up all the angles and solve for x:

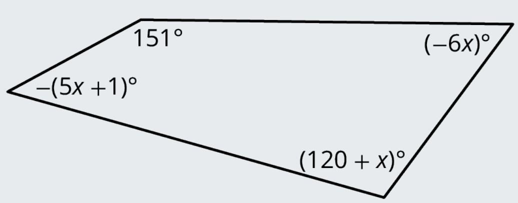
$$\begin{aligned} 5(x+7) + 120 + (6x+25) + 5(2x+5) + 5(3x-5) &= 540 \\ 5x + 6x + 10x + 15x + 180 &= 540 \\ 36x &= 360 \\ x &= 10 \end{aligned}$$

Step 3: We can then find the measure of each interior angle:

$$\begin{aligned} m\measuredangle A &= 5(10+7) = 85^\circ \\ m\measuredangle B &= 120^\circ \\ m\measuredangle C &= 6(10) + 25 = 85^\circ \\ m\measuredangle D &= 5(2 * 10 + 5) = 125^\circ \\ m\measuredangle E &= 5(3 * 10 - 5) = 125^\circ \end{aligned}$$

YOUR TURN 10.29

1. Find the sum of the measures of the interior angles and then find the measure of each interior angle in the figure shown.



An **exterior angle** of a regular polygon is an angle formed by extending a side length beyond the closed figure. The measure of an exterior angle of a regular polygon with n sides is found using the following formula:

FORMULA

To find the measure of an exterior angle of a regular polygon with n sides we use the formula

$$b = \frac{360^\circ}{n}.$$

In Figure 10.71, we have a regular hexagon $ABCDEF$. By extending the lines of each side, an angle is formed on the exterior of the hexagon at each vertex. The measure of each exterior angle is found using the formula, $b = \frac{360^\circ}{6} = 60^\circ$.

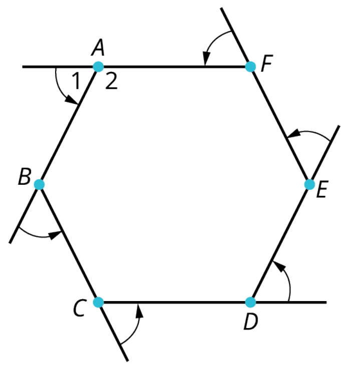


Figure 10.71 Exterior Angles

Now, an important point is that the **sum** of the exterior angles of a regular polygon with n sides equals 360°. This implies that when we multiply the measure of one exterior angle by the number of sides of the regular polygon, we should get 360°. For the example in Figure 10.71, we multiply the measure of each exterior angle, 60°, by the number of sides, six. Thus, the sum of the exterior angles is $6(60^\circ) = 360^\circ$.

EXAMPLE 10.30

Calculating the Sum of Exterior Angles

Find the sum of the measure of the exterior angles of the pentagon (Figure 10.72).

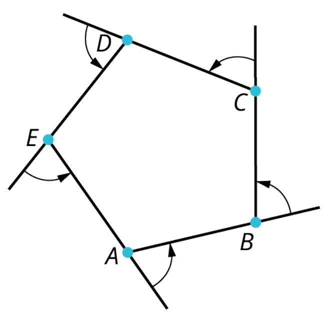


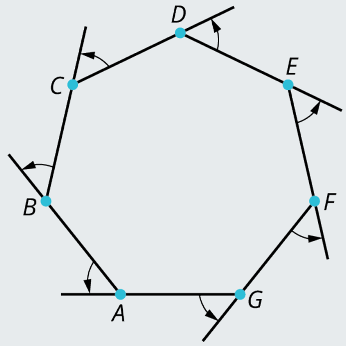
Figure 10.72

Solution

Each individual angle measures $\frac{360}{5} = 72^\circ$. Then, the sum of the exterior angles is $5(72^\circ) = 360^\circ$.

YOUR TURN 10.30

1. Find the sum of the measures of the exterior angles in the figure shown.



Circles and Circumference

The perimeter of a circle is called the **circumference**. To find the circumference, we use the formula $C = \pi d$, where d is the diameter, the distance across the center, or $C = 2\pi r$, where r is the radius.

FORMULA

The circumference of a circle is found using the formula $C = \pi d$, where d is the diameter of the circle, or $C = 2\pi r$, where r is the radius.

The radius is ½ of the diameter of a circle. The symbol $\pi = 3.141592654\ldots$ is the ratio of the circumference to the diameter. Because this ratio is constant, our formula is accurate for any size circle. See Figure 10.73.

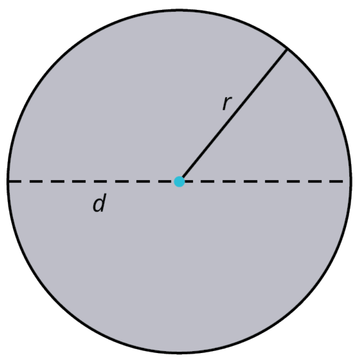


Figure 10.73 Circle Diameter and Radius

Let the radius be equal to 3.5 inches. Then, the circumference is

$$\begin{aligned} C &= 2\pi(3.5) \\ &= 21.99 \text{ in.} \end{aligned}$$

EXAMPLE 10.31

Finding Circumference with Diameter

Find the circumference of a circle with diameter 10 cm.

Solution

If the diameter is 10 cm, the circumference is $C = 10\pi = 31.42 \text{ cm}$.

YOUR TURN 10.31

1. Find the circumference of a circle with a radius of 2.25 cm.

EXAMPLE 10.32

Finding Circumference with Radius

Find the radius of a circle with a circumference of 12 in.

Solution

If the circumference is 12 in, then the radius is

$$12 = 2\pi r$$
$$\frac{12}{2\pi} = r = 1.91 \text{ in.}$$

YOUR TURN 10.32

1. Find the radius of a circle with a circumference of 15.71 cm.

EXAMPLE 10.33

Calculating Circumference for the Real World

You decide to make a trim for the window in Figure 10.74. How many feet of trim do you need to buy?

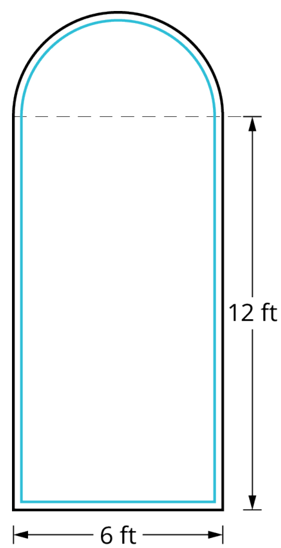


Figure 10.74

Solution

The trim will cover the 6 feet along the bottom and the two 12-ft sides plus the half circle on top. The circumference of a semicircle is ½ the circumference of a circle. The diameter of the semicircle is 6 ft. Then, the circumference of the semicircle would be $\frac{1}{2}\pi d = \frac{1}{2}\pi(6) = 3\pi \text{ ft} = 9.4 \text{ ft}$.

Therefore, the total perimeter of the window is $6 + 12 + 12 + 9.4 = 39.4 \text{ ft}$. You need to buy 39.4 ft of trim.

YOUR TURN 10.33

1. To make a trim for the window in the figure shown, you need the perimeter. How many feet of trim do you need to buy?

PEOPLE IN MATHEMATICS

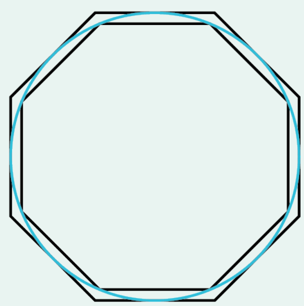
Archimedes

Figure 10.75 Archimedes (credit: “Archimedes” Engraving from the book Les vrais pourtraits et vies des hommes illustres grecz, latins et payens (1586)/Wikimedia Commons, Public Domain)

The overwhelming consensus is that Archimedes (287–212 BCE) was the greatest mathematician of classical antiquity, if not of all time. A Greek scientist, inventor, philosopher, astronomer, physicist, and mathematician, Archimedes flourished in Syracuse, Sicily. He is credited with the invention of various types of pulley systems and screw pumps based on the center of gravity. He advanced numerous mathematical concepts, including theorems for finding surface area and volume. Archimedes anticipated modern calculus and developed the idea of the “infinitely small” and the method of exhaustion. The method of exhaustion is a technique for finding the area of a shape inscribed within a sequence of polygons. The areas of the polygons converge to the area of the inscribed shape. This technique evolved to the concept of limits, which we use today.

One of the more interesting achievements of Archimedes is the way he estimated the number pi, the ratio of the circumference of a circle to its diameter. He was the first to find a valid approximation. He started with a circle having a diameter of 1 inch. His method involved drawing a polygon inscribed inside this circle and a polygon circumscribed around this circle. He knew that the perimeter of the inscribed polygon was smaller than the circumference of the circle, and the perimeter of the circumscribed polygon was larger than the circumference of the circle. This is shown in the drawing of an eight-sided polygon. He increased the number of sides of the polygon each time as he got closer to the real value of pi. The following table is an example of how he did this.

Sides	Inscribed Perimeter	Circumscribed Perimeter
4	2.8284	4.00
8	3.0615	3.3137
16	3.1214	3.1826
32	3.1365	3.1517
64	3.1403	3.1441



Archimedes settled on an approximation of $\pi \approx 3.1416$ after an iteration of 96 sides. Because pi is an irrational number, it cannot be written exactly. However, the capability of the supercomputer can compute pi to billions of decimal digits. As of 2002, the most precise approximation of pi includes 1.2 trillion decimal digits.

WHO KNEW?

The Platonic Solids

The Platonic solids (Figure 10.76) have been known since antiquity. A polyhedron is a three-dimensional object constructed with congruent regular polygonal faces. Named for the philosopher, Plato believed that each one of the solids is associated with one of the four elements: Fire is associated with the tetrahedron or pyramid, earth with the cube, air with the octahedron, and water with the icosahedron. Of the fifth Platonic solid, the dodecahedron, Plato said, “... God used it for arranging the constellations on the whole heaven.”

Tetrahedron	Cube	Octahedron	Dodecahedron	Icosahedron
Four faces	Six faces	Eight faces	Twelve faces	Twenty faces

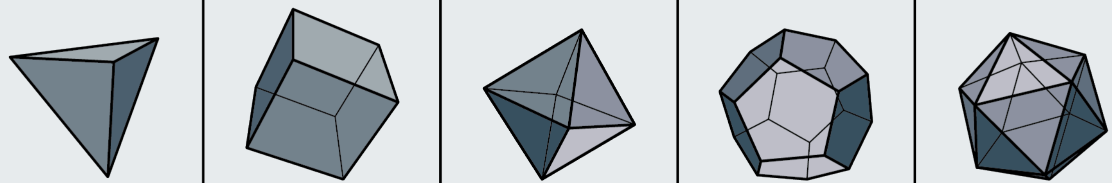
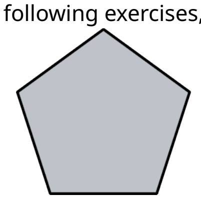


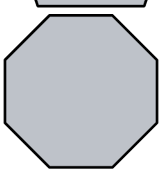
Figure 10.76 Platonic Solids

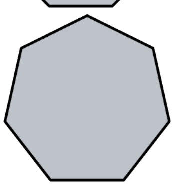
Plato believed that the combination of these five polyhedra formed all matter in the universe. Later, Euclid proved that exactly five regular polyhedra exist and devoted the last book of the *Elements* to this theory. These ideas were resuscitated by Johannes Kepler about 2,000 years later. Kepler used the solids to explain the geometry of the universe. The beauty and symmetry of the Platonic solids have inspired architects and artists from antiquity to the present.

Check Your Understanding

In the following exercises, identify the regular polygons.

22. 

23. 

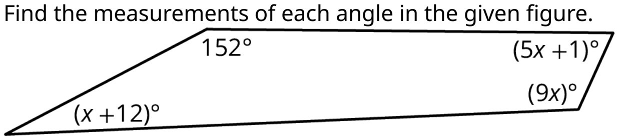
24. 

25. Find the perimeter of the regular pentagon with side length of 6 cm.

26. Find the sum of the interior angles of a regular hexagon.

27. Find the measure of each interior angle of a regular hexagon.

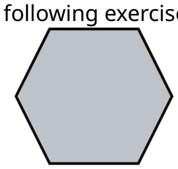
28. Find the measurements of each angle in the given figure.

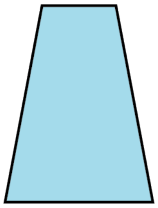


29. Find the circumference of the circle with radius equal to 3 cm.

SECTION 10.4 EXERCISES

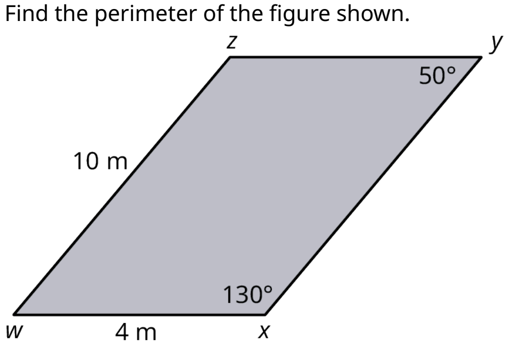
In the following exercises, identify the polygons.

1. 

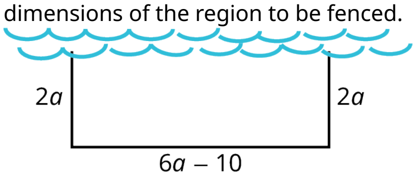
2. 

3. 

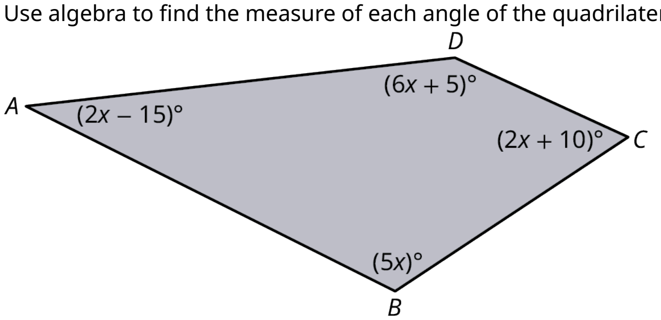
4. Find the perimeter of a regular hexagon with side length equal to 12 cm.
5. A regular quadrilateral has a perimeter equal to 72 in. Find the length of each side.
6. The perimeter of an equilateral triangle is 72 cm. Find the length of each side.
7. Find the dimensions of a rectangular region with perimeter of 34 m, where the shorter side is 13 less than twice the longer side. Let $x = \text{the longer side}$. Then, the shorter side is $(2x - 13)$.
8. Find the perimeter of the figure shown.



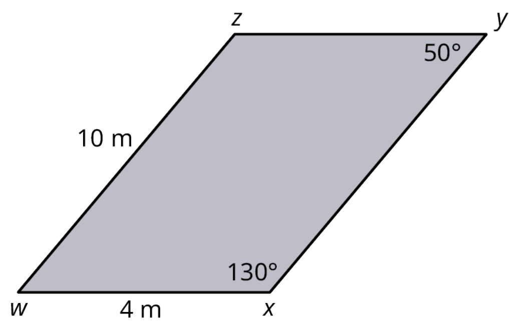
9. Find the perimeter of a fenced-in area where the length is 22 m, and the width is ½ of the length plus 3.
10. You have 140 ft of fencing to enclose a rectangular region that borders a river. You do not have to fence in the side that borders the river. The width is equal to $2a$, and the length is equal to six times the width less 10. Find the dimensions of the region to be fenced.



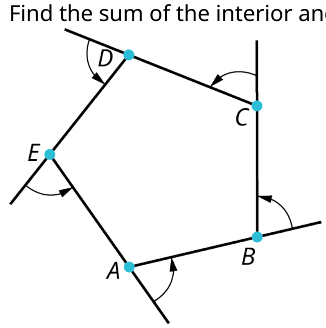
11. What is the measure of each interior angle of a regular hexagon?
12. What is the sum of the interior angles of a triangle?
13. Use algebra to find the measure of each angle of the quadrilateral shown.



14. Find the missing sides and angles of the parallelogram shown.



15. Find the sum of the interior and exterior angles of the regular pentagon shown.



16. What is the measure of each exterior angle of a regular hexagon?

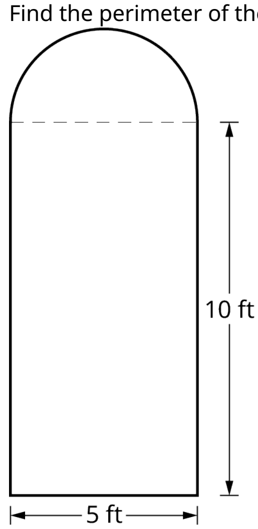
17. What is the sum of the exterior angles of a triangle?

18. What is the sum of the exterior angles of an octagon?

19. Find the circumference of a circle with radius 5.

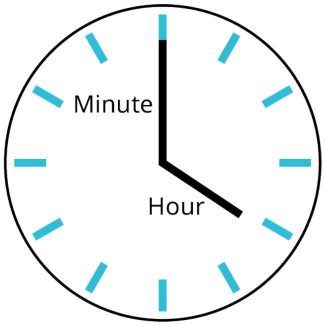
20. Find the circumference of a circle with a diameter of 7.

21. Find the perimeter of the window in the figure shown.

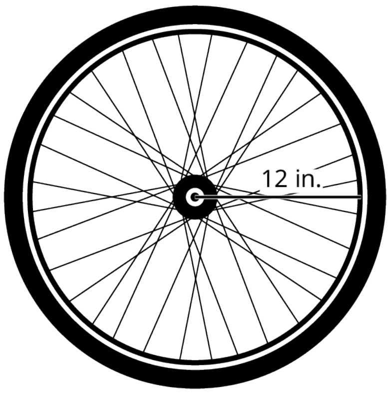


22. Find the circumference of a circle with a radius of 1.25 cm.

23. The hands of a clock vary in length. If the hour hand is 5 inches long and the minute hand is 7 inches long, how far does each hand travel in 12 hours?



24. A bicycle tire has a radius of 12 in. How many revolutions will the tire make if the bicycle travels approximately 377 ft?



25. When Chicago installed its Centennial Ferris wheel at Navy Pier in 1995, there was much discussion about how big it could be. The city wanted it to be as high as possible because the incredible views of the city would surely draw the tourists. It was decided that the height of the wheel could safely reach 150 ft. If the wheel begins 10 ft above ground level, what is the circumference of the Ferris wheel?

26. Find the perimeter of an equilateral triangle with side length equal to 21 cm.

27. What is the diameter of a circle whose circumference is 39.6 cm?