

motions of tessellations. Finally, we will introduce right-angle trigonometry and explore the Pythagorean Theorem.

10.1 Points, Lines, and Planes



Figure 10.2 The lower right-hand corner of *The School of Athens* depicts a figure representing Euclid illustrating to students how to use a compass on a small chalkboard. (credit: modification of work “School of Athens” by Raphael (1483–1520), Vatican Museums/Wikimedia, Public Domain)

Learning Objectives

After completing this section, you should be able to:

1. Identify and describe points, lines, and planes.
2. Express points and lines using proper notation.
3. Determine union and intersection of sets.

In this section, we will begin our exploration of geometry by looking at the basic definitions as defined by Euclid. These definitions form the foundation of the geometric theories that are applied in everyday life.

In *The Elements*, Euclid summarized the geometric principles discovered earlier and created an axiomatic system, a system composed of postulates. A *postulate* is another term for axiom, which is a statement that is accepted as truth without the need for proof or verification. There were no formal geometric definitions before Euclid, and when terms could not be defined, they could be described. In order to write his postulates, Euclid had to describe the terms he needed and he called the descriptions “definitions.” Ultimately, we will work with theorems, which are statements that have been proved and can be proved.

Points and Lines

The first definition Euclid wrote was that of a point. He defined a point as “that which has no part.” It was later expanded to “an indivisible location which has no width, length, or breadth.” Here are the first two of the five postulates, as they are applicable to this first topic:

1. **Postulate 1:** A straight line segment can be drawn joining any two points.
2. **Postulate 2:** Any straight line segment can be extended indefinitely in a straight line.

Before we go further, we will define some of the symbols used in geometry in [Figure 10.3](#):


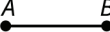
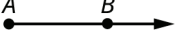


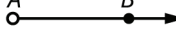

Symbol	Definition	Picture
•	Points are defined with capital letters, like point A	
\overline{AB}	A line segment from point A to point B	
\overrightarrow{AB}	A ray from point A in the direction of B	
\overleftarrow{AB}	A ray from point B in the direction of A	
\overleftrightarrow{AB}	A line that includes the points A and B goes off indefinitely in both directions	
$\overrightarrow{\circ AB}$	A half-line from, but not including, point A in the direction of point B	
$\overleftarrow{\circ AB}$	A half-line from, but not including, point B in the direction of A	

Figure 10.3 Basic Geometric Symbols for Points and Lines

From Figure 10.3, we see the variations in lines, such as line segments, rays, or half-lines. What is consistent is that two collinear points (points that lie on the same line) are required to form a line. Notice that a line segment is defined by its two endpoints showing that there is a definite beginning and end to a line segment. A ray is defined by two points on the line; the first point is where the ray begins, and the second point gives the line direction. A half-line is defined by two points, one where the line starts and the other to give direction, but an open circle at the starting point indicates that the starting point is not part of the half-line. A regular line is defined by any two points on the line and extends infinitely in both directions. Regular lines are typically drawn with arrows on each end.

EXAMPLE 10.1**Defining Lines**

For the following exercises, use this line (Figure 10.4).



Figure 10.4

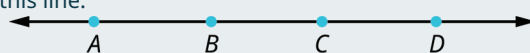
1. Define \overline{DE} .
2. Define F .
3. Define \overleftrightarrow{DF} .
4. Define \overline{EF} .

✓ Solution

1. The symbol \overline{DE} , two letters with a straight line above, refers to the line segment that starts at point D and ends at point E .
2. The letter F alone refers to point F .
3. The symbol \overleftrightarrow{DF} , two letters with a line above containing arrows on both ends, refers to the line that extends infinitely in both directions and contains the points D and F .
4. The symbol \overline{EF} , two letters with a straight line above, refers to the line segment that starts at point E and ends at point F .

> YOUR TURN 10.1

For the following exercises, use this line.



1. Define \overrightarrow{BD}
2. Define \overline{AB}
3. \overleftarrow{BA}
4. \overleftrightarrow{AD}

There are numerous applications of line segments in daily life. For example, airlines working out routes between cities, where each city's airport is a point, and the points are connected by line segments. Another example is a city map. Think about the intersection of roads, such that the center of each intersection is a point, and the points are connected by line segments representing the roads. See [Figure 10.5](#).

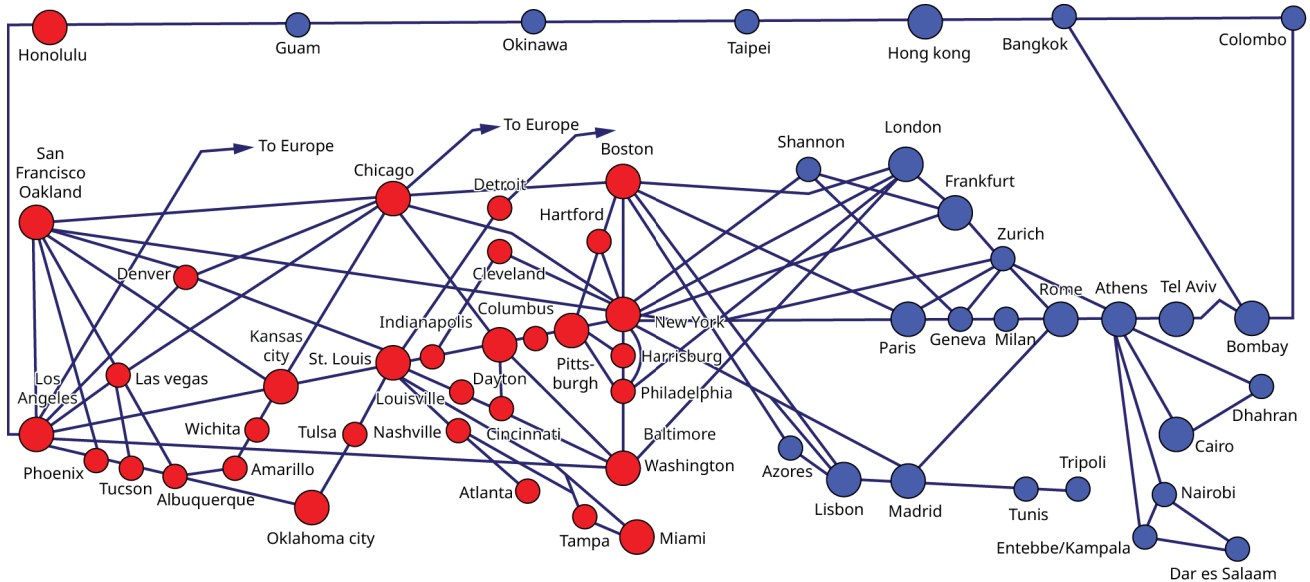


Figure 10.5 Air Line Routes

EXAMPLE 10.2

Determining the Best Route

View the street map ([Figure 10.6](#)) as a series of line segments from point to point. For example, we have vertical line segments \overline{AB} , \overline{BC} , and \overline{CD} on the right. On the left side of the map, we have vertical line segments \overline{HI} , \overline{FG} . The horizontal line segments are \overline{HA} , \overline{EI} , \overline{TE} , \overline{EB} , \overline{FC} , \overline{CF} , and \overline{GB} . There are two diagonal line segments, \overline{AE} and \overline{EF} . Assume that each location is on a corner and that you live next door to the library.

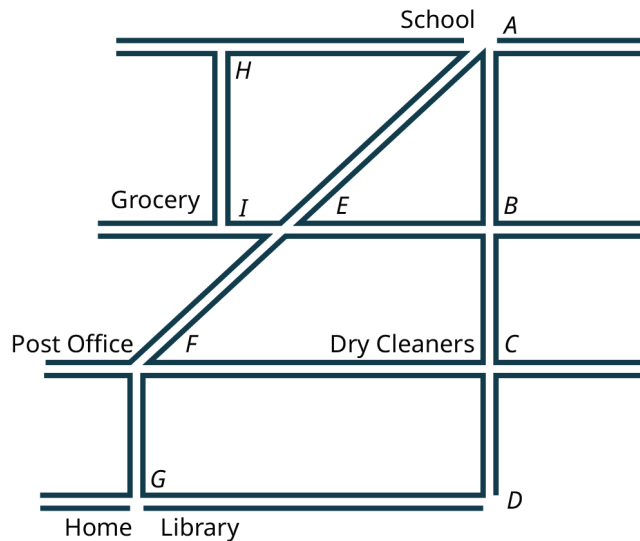


Figure 10.6 Street Map

1. Let's say that you want to stop at the grocery store on your way home from school. Come up with three routes you might take to do your errand and then go home. In other words, name the three ways by the line segments in the order you would walk, and which way do you think would be the most efficient route?
2. How about stopping at the library after school? Name four ways you might travel to the library and which way do you think is the most efficient?

3. Suppose you need to go to the post office and the dry cleaners on your way home from school. Name three ways you might walk to do your errands and end up at home. Which way do you think is the most efficient way to walk, get your errands done, and go home?

Solution

- From school to the grocery store and home: first way $\overline{AH}, \overline{HI}, \overline{IE}, \overline{EF}, \overline{FG}$; second way $\overline{AB}, \overline{BE}, \overline{EI}, \overline{IE}, \overline{EF}, \overline{FG}$; third way $\overline{AE}, \overline{EI}, \overline{IE}, \overline{EF}, \overline{FG}$. It seems that the third way is the most efficient way.
- From school to the library: first way $\overline{AE}, \overline{EF}, \overline{FG}$; second way $\overline{AB}, \overline{BC}, \overline{CD}, \overline{DG}$; third way $\overline{AB}, \overline{BE}, \overline{EF}, \overline{FG}$; fourth way $\overline{AB}, \overline{BC}, \overline{CF}, \overline{FG}$. The first way should be the most efficient way.
- From school to the post office or dry cleaners to home: first way $\overline{AE}, \overline{EF}, \overline{FC}, \overline{CD}, \overline{DG}$; second way $\overline{AB}, \overline{BE}, \overline{EF}, \overline{FC}, \overline{CD}, \overline{DG}$; third way $\overline{AB}, \overline{BC}, \overline{CF}, \overline{FG}$. The third way would be the most efficient way.

YOUR TURN 10.2

- Using the street map in Figure 10.6, find two ways you would stop at the dry cleaners and the grocery store after school on your way home.

Parallel Lines

Parallel lines are lines that lie in the same plane and move in the same direction, but never intersect. To indicate that the line l_1 and the line l_2 are parallel we often use the symbol $l_1 \parallel l_2$. The distance d between parallel lines remains constant as the lines extend infinitely in both directions. See [Figure 10.7](#).

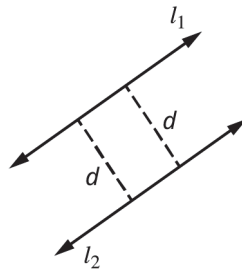


Figure 10.7 Parallel Lines

Perpendicular Lines

Two lines that intersect at a 90° angle are **perpendicular lines** and are symbolized by \perp . If l_1 and l_2 are perpendicular, we write $l_1 \perp l_2$. When two lines form a right angle, a 90° angle, we symbolize it with a little square \square . See [Figure 10.8](#).

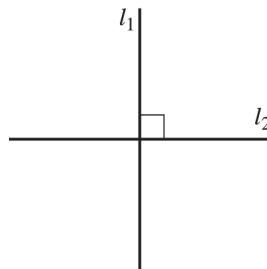


Figure 10.8 Perpendicular Lines

EXAMPLE 10.3

Identifying Parallel and Perpendicular Lines

Identify the sets of parallel and perpendicular lines in [Figure 10.9](#).

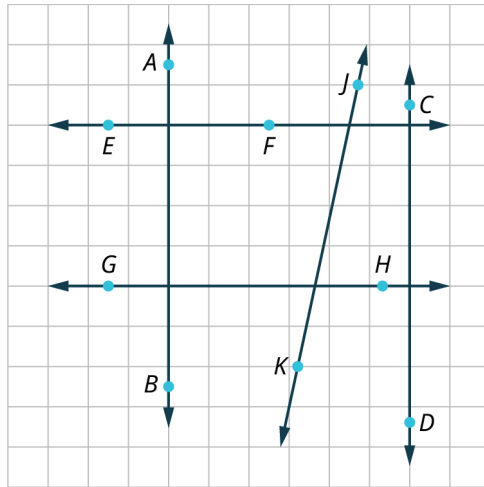


Figure 10.9

✓ **Solution**

Drawing these lines on a grid is the best way to distinguish which pairs of lines are parallel and which are perpendicular. Because they are on a grid, we assume all lines are equally spaced across the grid horizontally and vertically. The grid also tells us that the vertical lines are parallel and the horizontal lines are parallel. Additionally, all intersections form a 90° angle. Therefore, we can safely say the following:

$\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$, the line containing the points A and B is parallel to the line containing the points C and D .

$\overleftrightarrow{EF} \parallel \overleftrightarrow{GH}$, the line containing the points E and F is parallel to the line containing the points G and H .

$\overleftrightarrow{AB} \perp \overleftrightarrow{EF}$, the line containing the points A and B is perpendicular to the line containing the points E and F . We know this because both lines trace grid lines, and intersecting grid lines are perpendicular.

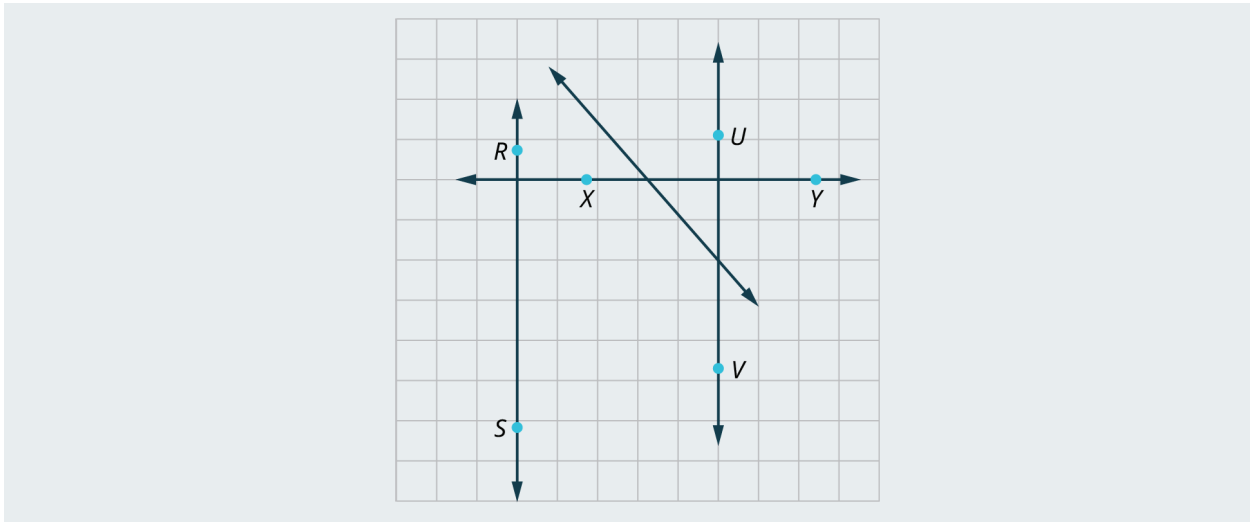
We can also state that $\overleftrightarrow{AB} \perp \overleftrightarrow{GH}$; the line containing the points A and B is perpendicular to the line containing the points G and H because both lines trace grid lines, which are perpendicular by definition.

We also have $\overleftrightarrow{CD} \perp \overleftrightarrow{EF}$; the line containing the points C and D is perpendicular to the line containing the points E and F because both lines trace grid lines, which are perpendicular by definition.

Finally, we see that $\overleftrightarrow{CD} \perp \overleftrightarrow{GH}$; the line containing the points C and D is perpendicular to the line containing the points G and H because both lines trace grid lines, which are perpendicular by definition.

> **YOUR TURN 10.3**

1. Identify the sets of parallel and perpendicular lines in the given figure.



Defining Union and Intersection of Sets

Union and intersection of sets is a topic from set theory that is often associated with points and lines. So, it seems appropriate to introduce a mini-version of set theory here. First, a **set** is a collection of objects joined by some common criteria. We usually name sets with capital letters. For example, the set of odd integers between 0 and 10 looks like this: $A = \{1, 3, 5, 7, 9\}$. When it involves sets of lines, line segments, or points, we are usually referring to the **union** or **intersection** of set.

The union of two or more sets contains all the elements in either one of the sets or elements in all the sets referenced, and is written by placing this symbol \cup in between each of the sets. For example, let set $A = \{1, 2, 3\}$, and let set $B = \{4, 5, 6\}$. Then, the union of sets A and B is $A \cup B = \{1, 2, 3, 4, 5, 6\}$.

The intersection of two or more sets contains only the elements that are common to each set, and we place this symbol \cap in between each of the sets referenced. For example, let's say that set $A = \{1, 3, 5\}$, and let set $B = \{5, 7, 9\}$. Then, the intersection of sets A and B is $A \cap B = \{5\}$.

EXAMPLE 10.4

Defining Union and Intersection of Sets

Use the line (Figure 10.10) for the following exercises. Draw each answer over the main drawing.

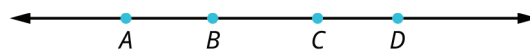


Figure 10.10

1. Find $\overrightarrow{BD} \cap \overleftarrow{CA}$.
2. Find $\overline{AB} \cup \overline{AD}$.
3. Find $\overleftrightarrow{AD} \cup \overleftrightarrow{BC}$.
4. Find $\overleftrightarrow{AD} \cap \overleftrightarrow{BC}$.
5. Find $\overleftarrow{BA} \cap \overleftarrow{CD}$.

✓ Solution

1. Find $\overrightarrow{BD} \cap \overleftarrow{CA}$. This is the intersection of the ray \overrightarrow{BD} and the ray \overleftarrow{CA} . Intersection includes only the elements that are common to both lines. For this intersection, only the line segment \overline{BC} is common to both rays. Thus, $\overrightarrow{BD} \cap \overleftarrow{CA} = \overline{BC}$.

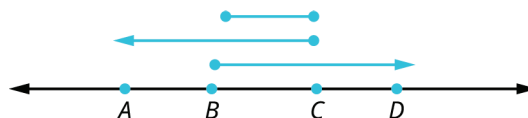


Figure 10.11

2. Find $\overline{AB} \cup \overline{AD}$. The problem is asking for the union of two line segments, \overline{AB} and \overline{AD} . Union includes all elements

in the first line and all elements in the second line. Since \overline{AB} is part of \overline{AD} , $\overline{AB} \cup \overline{AD} = \overline{AD}$.



Figure 10.12

3. Find $\overleftrightarrow{AD} \cup \overline{BC}$. This is the union of the line \overleftrightarrow{AD} with the line segment \overline{BC} . As the line segment \overline{BC} is included on the line \overleftrightarrow{AD} , then the union of these two lines equals the line \overleftrightarrow{AD} .



Figure 10.13

4. Find $\overleftrightarrow{AD} \cap \overline{BC}$. The intersection of the line \overleftrightarrow{AD} with line segment \overline{BC} is the set of elements common to both lines. In this case, the only element in common is the line segment \overline{BC} .

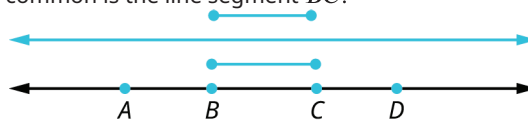


Figure 10.14

5. Find $\overleftrightarrow{AB} \cap \overline{CD}$. This is the intersection of the ray \overleftrightarrow{AB} and the line segment \overline{CD} . Intersection includes elements common to both lines. There are no elements in common. The intersection yields the empty set, as shown in [Figure 10.15](#). Therefore, $\overleftrightarrow{AB} \cap \overline{CD} = \emptyset$.

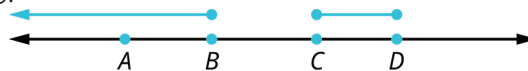
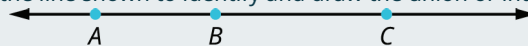


Figure 10.15

> YOUR TURN 10.4

For the following exercises, use the line shown to identify and draw the union or intersection of sets.

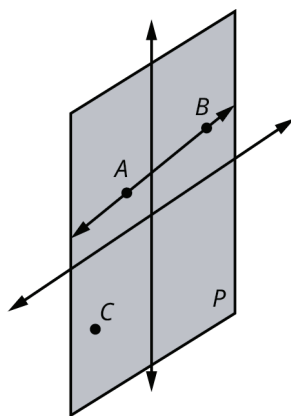


1. $\overline{AB} \cap \overline{BC}$
2. $\overline{BC} \cup \overline{CA}$
3. $\overline{BC} \cap \overline{AC}$

Planes

A **plane**, as defined by Euclid, is a “surface which lies evenly with the straight lines on itself.” A plane is a two-dimensional surface with infinite length and width, and no thickness. We also identify a plane by three noncollinear points, or points that do not lie on the same line. Think of a piece of paper, but one that has infinite length, infinite width, and no thickness. However, not all planes must extend infinitely. Sometimes a plane has a limited area.

We usually label planes with a single capital letter, such as Plane P , as shown in [Figure 10.16](#), or by all points that determine the edges of a plane. In the following figure, Plane P contains points A and B , which are on the same line, and point C , which is not on that line. By definition, P is a plane. We can move laterally in any direction on a plane.

Figure 10.16 Plane P

One way to think of a plane is the Cartesian coordinate system with the x -axis marked off in horizontal units, and y -axis marked off in vertical units. In the Cartesian plane, we can identify the different types of lines as they are positioned in the system, as well as their locations. See [Figure 10.17](#).

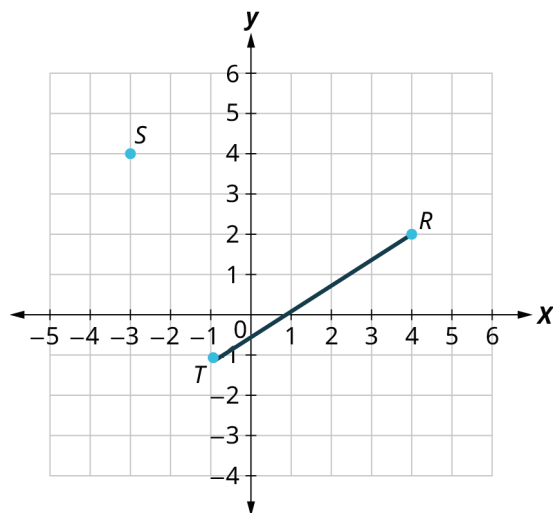


Figure 10.17 Cartesian Coordinate Plane

This plane contains points S , T , and R . Points T and R are collinear and form a line segment. Point S is not on that line segment. Therefore, this represents a plane.

To give the location of a point on the Cartesian plane, remember that the first number in the ordered pair is the horizontal move and the second number is the vertical move. Point R is located at $(4, 2)$; point S is located at $(-3, 4)$; and point T is located at $(-1, -1)$. We can also identify the line segment as \overline{TR} .

Two other concepts to note: Parallel planes do not intersect and the intersection of two planes is a straight line. The equation of that line of intersection is left to a study of three-dimensional space. See [Figure 10.18](#).

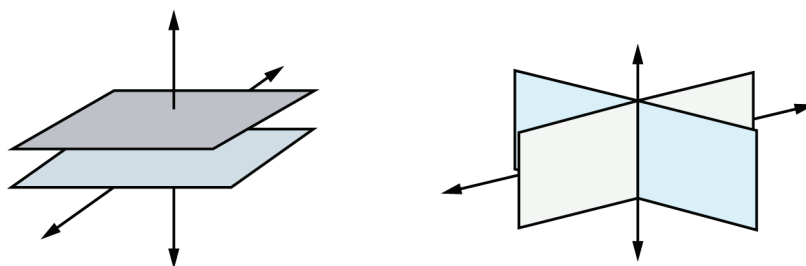


Figure 10.18 Parallel and Intersecting Planes

To summarize, some of the properties of planes include:

- Three points including at least one noncollinear point determine a plane.

- A line and a point not on the line determine a plane.
- The intersection of two distinct planes is a straight line.

EXAMPLE 10.5**Identifying a Plane**

For the following exercises, refer to [Figure 10.19](#)

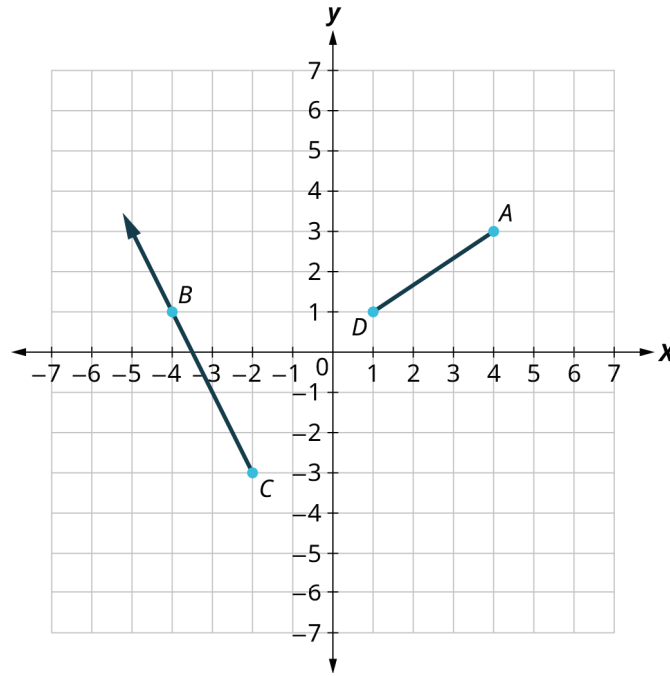


Figure 10.19

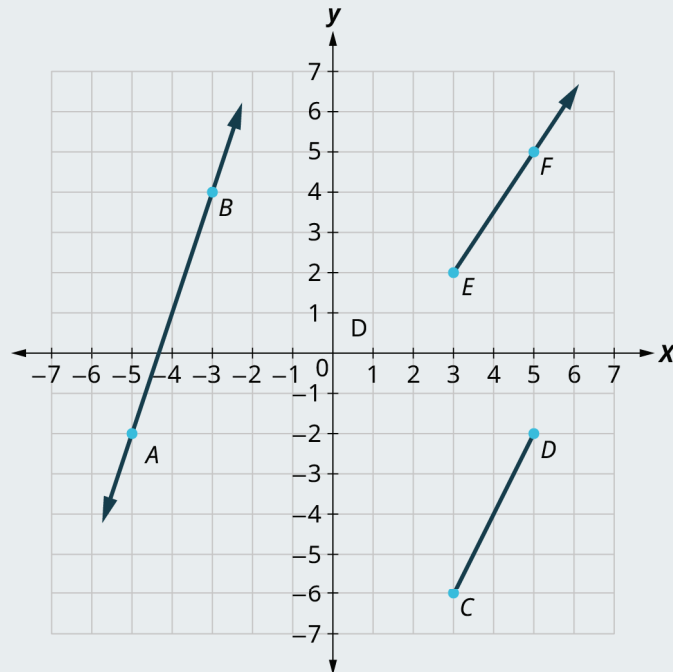
1. Identify the location of points A , B , C , and D .
2. Describe the line from point A to point D .
3. Describe the line from point C containing point B .
4. Does this figure represent a plane?

✓ **Solution**

1. Point A is located at $(4, 3)$; point B is located at $(-4, 1)$; point C is located at $(-2, -3)$; point D is located at $(1, 1)$.
2. The line from point A to point D is a line segment \overline{AD} .
3. The line from point C containing point B is a ray \overrightarrow{CB} starting at point C in the direction of B .
4. Yes, this figure represents a plane because it contains at least three points, points A and D form a line segment, and neither point B nor point C is on that line segment.

> **YOUR TURN 10.5**

For the following exercises, refer to the given figure.



1. Identify the location of points A , B , C , D , E , and F .
2. Describe the line that includes point A and point B .
3. Describe the line from E to point F .
4. Describe the line from C to point D .
5. Does this figure represent a plane?

EXAMPLE 10.6

Intersecting Planes

Name two pairs of intersecting planes on the shower enclosure illustration (Figure 10.20).

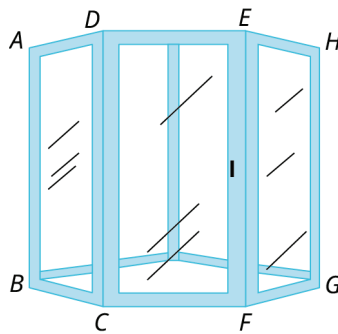


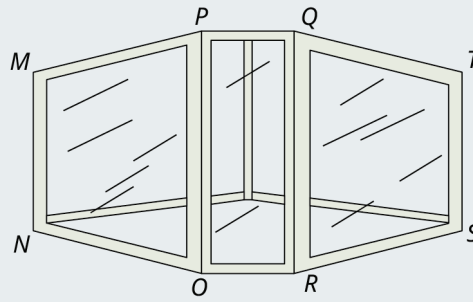
Figure 10.20

✓ Solution

The plane $ABCD$ intersects plane $CDEF$, and plane $CDEF$ intersects plane $EFGH$.

> YOUR TURN 10.6

1. Name two pairs of intersecting planes on the shower enclosure shown.



PEOPLE IN MATHEMATICS

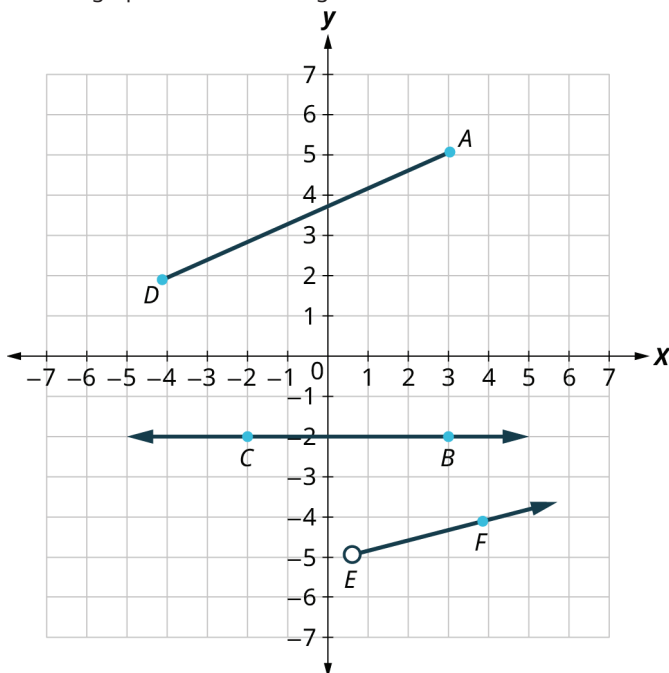
Plato

Part of a remarkable chain of Greek mathematicians, Plato (427–347 BC) is known as the teacher. He was responsible for shaping the development of Western thought perhaps more powerfully than anyone of his time. One of his greatest achievements was the founding of the Academy in Athens where he emphasized the study of geometry. Geometry was considered by the Greeks to be the “ultimate human endeavor.” Above the doorway to the Academy, an inscription read, “Let no one ignorant of geometry enter here.”

The curriculum of the Academy was a 15-year program. The students studied the exact sciences for the first 10 years. Plato believed that this was the necessary foundation for preparing students’ minds to study relationships that require abstract thinking. The next 5 years were devoted to the study of the “dialectic.” The dialectic is the art of question and answer. In Plato’s view, this skill was critical to the investigation and demonstration of innate mathematical truths. By training young students how to prove propositions and test hypotheses, he created a culture in which the systematic process was guaranteed. The Academy was essentially the world’s first university and held the reputation as the ultimate center of learning for more than 900 years.

Check Your Understanding

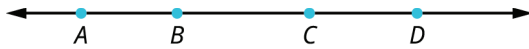
Use the graph for the following exercises.



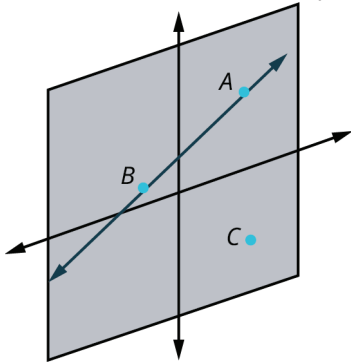
1. Identify the type of line containing point D and point A .

- Identify the type of line containing points C and B .
- Identify the type of line that references point E and contains point F .

Use the line for the following exercises.

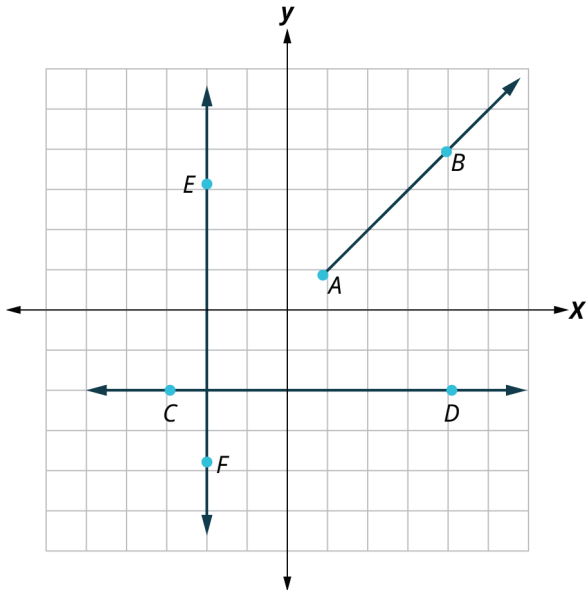


- Determine $\overline{AB} \cup \overline{BD}$.
- Determine $\overleftrightarrow{BD} \cap \overleftrightarrow{BC}$.
- Determine $\overleftrightarrow{BA} \cup \overleftrightarrow{BD}$.
- How can you determine whether two lines are parallel?
- How can you determine whether two lines are perpendicular?
- Determine if the illustration represents a plane.



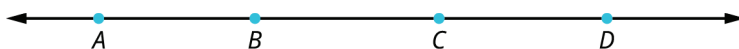
SECTION 10.1 EXERCISES

Use this graph for the following exercises.



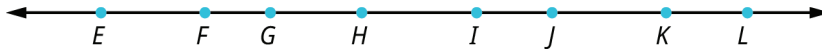
- Identify the kind of line passing through the points A and B .
- Identify the kind of line passing through the points C and D .
- Identify all sets of parallel lines.
- Identify all sets of perpendicular lines.

Use the line for the following exercises.



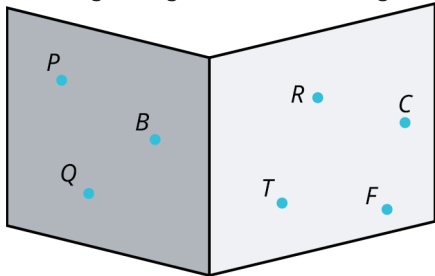
5. Find $\overleftrightarrow{BA} \cup \overleftrightarrow{BC}$.
6. Find $\overleftrightarrow{AD} \cup \overleftrightarrow{CD}$.
7. Find $\overleftrightarrow{AC} \cap \overleftrightarrow{BD}$.
8. Find $\overleftrightarrow{CA} \cap \overleftrightarrow{BD}$.
9. Find $\overleftrightarrow{AB} \cup \overleftrightarrow{BD}$.
10. Find $\overleftrightarrow{BC} \cup \overleftrightarrow{BD}$.

For the following exercises, reference this line.



11. Name the points in the set $\overleftrightarrow{FH} \cap \overleftrightarrow{GH}$.
12. Name the points contained in the set $\overleftrightarrow{HL} \cap \overleftrightarrow{JL}$.
13. Name the points in the set $\overleftrightarrow{EL} \cup \overleftrightarrow{GH}$.
14. Name the points in the set $\overleftrightarrow{GE} \cup \overleftrightarrow{KL}$.
15. Name the points in the set $\overleftrightarrow{GE} \cap \overleftrightarrow{KL}$.
16. Name the points in the set $\overleftrightarrow{HK} \cap \overleftrightarrow{JF}$.
17. Name the points in the set $\overleftrightarrow{GL} \cap K$.
18. Name the points in the set $\overleftrightarrow{HE} \cup \overleftrightarrow{KL}$.

Use the given figure for the following exercises.



19. Describe the lighter plane (right) in the drawing.
20. Describe the darker plane (left) in the drawing.